A STUDY ON L-FUZZY VECTOR SUBSPACES
AND ITS FUZZY DIMENSION

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ABSTRACT: This paper gives the definition of L-fuzzy vector subspace and defining its dimension by an L-fuzzy natural number. It is proved that for a finite dimensional L-fuzzy vector subspace, the intersection of two L-fuzzy vector subspace is also a L-fuzzy vector subspace and also the inequality \( \dim(\mathbb{E}_1 + \mathbb{E}_2) + \dim(\mathbb{E}_1 \cap \mathbb{E}_2) = \dim\mathbb{E}_1 + \dim\mathbb{E}_2 \) holds without any restricted conditions.

INTRODUCTION

Fuzzy vector space was introduced by Katsaras and Liu. The dimension of a fuzzy vector space is defined as a n-tuple by Lowen. The study of fuzzy vector spaces started as early as 1977. A fuzzy subset of a non-empty set \( S \) is a function from \( S \) into \([0,1]\). Let \( A \) denote a fuzzy subspace of \( V \) over a fuzzy subfield \( K \) of \( F \) and let \( X \) denote a fuzzy subset of \( V \) such that \( X \subseteq A \). Let \( \langle X \rangle \) denote the intersection of all fuzzy subspaces of \( V \) over \( K \) that contain \( X \) and are contained in \( A \).

PRELIMINARIES

Consider the set \( X \) and completely distributive lattice \( L \). Let the power set of \( X \) be \( 2^X \) and the set of all \( L \)-fuzzy sets on \( X \) be \( L^X \) respectively. For any \( A \subseteq X \), the cardinality of \( A \) be denoted by \(|A|\). An element \( L \) is called a prime element if \( a \geq b \wedge c \) implies \( a \geq b \) or \( a \geq c \) and an element \( L \) is called co-prime if \( a \leq b \vee c \) implies \( a \leq b \) or \( a \leq c \).

The set of non-unit prime elements in \( L \) is denoted by \( P(L) \) and the set of non-zero co-prime elements in \( L \) is denoted by \( J(L) \). The binary relation \( < \) is defined by for all \( a, b \in L \), \( a < b \) if and only if for every subset \( D \subseteq L \) with \( a \leq d \), the relation \( b \leq \sup D \) is possible only when \( d \in D \) with \( a \leq d \). The greatest minimal family of \( b \) is denoted by \( \beta(b) = \{ a \in L : a < b \} \) and \( \beta^*(b) = \beta(b) \cap J(L) \). Moreover for \( b \in L \) we define \( \alpha(b) = \{ a \in L : a < b \} \) and \( \alpha^*(b) = \alpha(b) \cap P(L) \). In a completely distributive lattice \( L \), there exist \( \alpha(b) \) and \( \beta(b) \) for each \( b \in L \) and \( b = \lor \beta(b) = \land \alpha(b) \).

Let \( \mathbb{N}(L) \) denotes the \( L \)-fuzzy natural number and the relation of \( \alpha \)-cut sets are defined as follows

For any \( \lambda, \mu \in \mathbb{N}(L) \), \( a \in L \),

(i) \( (\lambda + \mu)(a) \subseteq \lambda(a) + \mu(a) \subseteq (\lambda + \mu)[a] \);

(ii) \( (\lambda + \mu)^\alpha(a) \subseteq \lambda^\alpha(a) + \mu^\alpha(a) \subseteq (\lambda + \mu)^\alpha[a] \);

(iii) For any \( \lambda, \mu \in \mathbb{N}(L) \) and \( a \in P(L) \) implies \( (\lambda + \mu)^\alpha = \lambda^\alpha(a) + \mu^\alpha(a) \).

1. L-FUZZY VECTOR SUBSPACES

DEFINITION 1.1

L-FUZZY VECTOR SUBSPACE

L-Fuzzy Vector Subspace (LFVS) is a pair \( \overline{E} = (E, \mu) \) where \( E \) is a vector space on field \( F \), \( \mu : E \rightarrow L \) is a map with the property that for any \( x, y \in E \) and \( k, l \in F \) such that \( \mu(kx + ly) \geq \mu(x) \mu(y) \).

When \( L = [0,1] \) then L-Fuzzy Vector Subspace becomes fuzzy vector subspace. Let \( \overline{E} = (E, \mu) \) be a member of LFVS then

\[
\overline{E}^{(\alpha)} = \{ x \in E : \mu(x) \geq a \} , \quad \overline{E}^{(\beta)} = \{ x \in E : \mu(x) \leq a \} ,
\]

\[
\overline{E}^{(\alpha)} = \{ x \in E : \mu(x) \leq a \} , \quad \overline{E}^{(\beta)} = \{ x \in E : \mu(x) \geq a \} .
\]
THEOREM: 1.1

Let $E$ be a vector space, $\mu \in L^E$ and $\tilde{E} = (E, \mu)$ then the following statements are equivalent.

(i) $\tilde{E}$ is an $L$-fuzzy vector subspace. (ii) For all $a \in L$, $\tilde{E}_{[a]}$ is a vector space.

(iii) For all $a \in J(L)$, $\tilde{E}_{[a]}$ is a vector space. (iv) For all $a \in L$, $\tilde{E}^{[a]}$ is a vector space.

(v) For all $a \in P(L)$, $\tilde{E}^{[a]}$ is a vector space. (vi) For all $a \in P(L)$, $\tilde{E}^{[a]}$ is a vector space.

PROOF:

It is enough if we prove $1 \iff 4$ and $1 \iff 6$

(i) Assume that $\tilde{E}$ is an $L$-fuzzy vector subspace

Suppose that $x, y \in \tilde{E}_{[a]}$ then $a \not\in \alpha(\mu(x))$ and $a \not\in \alpha(\mu(y))$

i.e. $a \not\in \alpha(\mu(x)) \cup \alpha(\mu(y)) = \alpha(\mu(x) \Lambda \mu(y))$

then $\alpha(\mu(x) \Lambda \mu(y)) \supseteq \alpha(\mu(kx+ly))$

We have $a \not\in \alpha(\mu(kx+ly))$

Hence $kx+ly \in \tilde{E}_{[a]}$

Therefore $\tilde{E}_{[a]}$ is a vector space.

(ii) Suppose that $x, y \in \tilde{E}^{[a]}$ then $\mu(x) \not\leq a$ and $\mu(y) \not\leq a$

Since $a \in P(L)$ then $\mu(x) \Lambda \mu(y) \not\leq a$ (Since $\tilde{E} = (E, \mu)$ is an LFVS)

That is $\mu(kx+ly) \not\leq a$

Implies $kx+ly \in \tilde{E}^{[a]}$

Therefore $\tilde{E}^{[a]}$ is a vector space.

Assume $x, y \in E$ and $k, l \in F$ then

$kx+ly \in \tilde{E}^{[a]}$ if and only if $x \in \tilde{E}^{[a]}$ and $y \in \tilde{E}^{[a]}$ (Since $\tilde{E}^{[a]}$ is a vector space)

We have $\mu(kx+ly) = \Lambda a \in P(L) \ (a \Lambda \tilde{E}^{[a]})(kx+ly)$
\[ = \bigwedge_{a \in P(L)} (a \vee \bar{E}^{(a)} (x) \wedge \bar{E}^{(a)} (y))) \]

\[ = \left( \bigwedge_{a \in P(L)} (a \vee \bar{E}^{(a)} (x)) \right) \wedge \left( \bigwedge_{a \in P(L)} (a \vee \bar{E}^{(a)} (y)) \right) \]

\[ = \mu(x) \wedge \mu(y) \]

Therefore \( \bar{E} \) is an L-fuzzy vector subspace. Therefore \( 1 \Leftrightarrow 6 \)

Hence the Theorem.

**THEOREM 1.2**

Let \( V \) be a vector space, \( \mu: E \rightarrow L \) is a map and for all \( a,b \in L, \beta (a \Lambda b) = \beta(a) \cap \beta(b) \) then the following statements are equivalent:

1. \( \bar{E} \) is an L-fuzzy vector subspace.
2. For all \( a \in L, \bar{E}^{(a)} \) is a vector space.

**PROOF:**

Assume \( \bar{E} \) is an L-fuzzy vector subspace.

Suppose that \( x,y \in \bar{E}^{(a)} \) then \( a \in \beta(\mu(x)) \) and \( a \in \beta(\mu(y)) \)

i.e \( a \in \beta(\mu(x)) \cap \beta(\mu(y)) \)

Since for all \( a,b \in L, \beta(a \Lambda b) = \beta(a) \cap \beta(b) \) and \( \bar{E} \) is an L-fuzzy vector subspace

i.e \( a \in \beta(\mu(x)) \cap \beta(\mu(y)) \subseteq \beta(\mu(ax+by)) \)

\[ \Rightarrow ax+by \in \bar{E}^{(a)} \]

Therefore \( \bar{E}^{(a)} \) is a vector space.

Next assume that for all \( a \in L, \bar{E}^{(a)} \) is a vector space.

Let \( x,y \in E \) and \( k,l \in F \) then \( kx+ly \in \bar{E}^{(a)} \) if and only if \( x \in \bar{E}^{(a)} \) and \( y \in \bar{E}^{(a)} \) (Since \( \bar{E}^{(a)} \) is a vector space)

We have \( \mu (kx+ly) = \bigvee_{a \in L} (a \Lambda \bar{E}^{(a)} (x+y)) \)

\[ = \bigvee_{a \in L} (a \Lambda (\bar{E}^{(a)} (x) \Lambda \bar{E}^{(a)} (y))) \]

\[ = (\bigvee_{a \in L} (a \Lambda (\bar{E}^{(a)} (x))) \Lambda (\bigvee_{a \in L} (a \Lambda (\bar{E}^{(a)} (y)))) \]

\[ = \mu(x) \wedge \mu(y) \]

Therefore \( \bar{E} \) is an L-fuzzy vector subspace.

Therefore the above two statements are equivalent.

**DEFINITION 1.2**

Let \( \bar{E}_1 = (E, \mu_1) \) and \( \bar{E}_2 = (E, \mu_2) \) be two fuzzy vector subspaces on \( E \). The intersection of \( \bar{E}_1 \) and \( \bar{E}_2 \) is defined as \( \bar{E}_1 \cap \bar{E}_2 = (E, \mu_1 \Lambda \mu_2) \) and the sum of \( \bar{E}_1 \) and \( \bar{E}_2 \) is defined as \( \bar{E}_1 + \bar{E}_2 = (E, \mu_1 + \mu_2) \)

Where \( \mu_1 + \mu_2 \) is defined as for all \( x \in E \), \( (\mu_1 + \mu_2)(x) = \bigvee (\mu_1(x_1) \Lambda \mu_2(x_2)) \)
DEFINITION 1.3

Let \( E_1 = (E, \mu_1) \) and \( E_2 = (E, \mu_2) \) be two members on LFVS and \( E = E_1 \oplus E_2 \) be the direct sum of \( E_1 \) and \( E_2 \) defined as \( E_i \oplus E_2 = (E, \mu_i \oplus \mu_2) \) where \( \mu_i \oplus \mu_2 \) is defined as for all \( x \in E \), \( x = x_1 \oplus x_2 \), \( x_i \in E_i \), \( i = 1, 2 \).

\[
(\mu_1 \oplus \mu_2)(x) = (\mu_1 \oplus \mu_2)(x_1 \oplus x_2) = \mu_1(x_1) \land \mu_2(x_2).
\]

THEOREM 1.3

Let \( E_1 = (E, \mu_1) \) and \( E_2 = (E, \mu_2) \) be two members on LFVS on \( E \) we have

(i) \( E_1 \cap E_2 \) is a member of LFVS on \( E \).

(ii) \( E_1 + E_2 \) is a member of LFVS on \( E \).

PROOF:

Given \( E_1 \) and \( E_2 \) be two members on LFVS then \( \mu_1(\alpha x + \beta y) \geq \mu_1(\alpha x) \land \mu_1(\beta y) \) and \( \mu_2(\alpha x + \beta y) \geq \mu_2(\alpha x) \land \mu_2(\beta y) \).

To prove \( E_1 \cap E_2 \) is a member of LFVS on \( E \)

\[
E_1 \cap E_2 = (E, \mu_1 \land \mu_2) \quad \text{(By definition 3.2)}
\]

Consider \( (E, \mu_1 \land \mu_2) = \mu_1(\alpha x + \beta y) \land \mu_2(\alpha x + \beta y) \geq (\mu_1(\alpha x) \land \mu_1(\beta y) \land \mu_2(\alpha x) \land \mu_2(\beta y)). \)

Therefore \( E_1 \cap E_2 \) is a member of LFVS on \( E \).

Similarly we can prove \( E_1 + E_2 \) is a member of LFVS on \( E \).

THEOREM 1.4

Let \( E_1 = (E, \mu_1) \) and \( E_2 = (E, \mu_2) \) be two members on LFVS on \( E \). Suppose that for any \( a, b \in L \), we have \( \beta(a \land b) = \beta(a) \cap \beta(b) \) then

(1) \( (E_1 \cap E_2)(\alpha) = (E_1)(\alpha) \cap (E_2)(\alpha) \)

(2) \( (E_1 + E_2)(\alpha) = (E_1)(\alpha) \cup (E_2)(\alpha) \).

2. FUZZY DIMENSION OF L-FUZZY VECTOR SUBSPACES

DEFINITION 2.1

Let \( \mathbb{N}(L) \) be the family of L-fuzzy natural number. The map \( \dim : \text{LFVS} \rightarrow \mathbb{N}(L) \) is defined by \( \dim E(n) = \lor \{a \in L : \dim E(a) \geq n\} \) is called the L-fuzzy dimensional function of the L-fuzzy vector subspace \( E \), it is an fuzzy natural number.

Also \( \dim E(n) = \lor \{a \in L : \dim E(a) \geq n\} \).

THEOREM 2.1

Let \( E_1 = (E, \mu_1) \) and \( E_2 = (E, \mu_2) \) be two L-fuzzy vector subspaces then the following equalities holds

\( \dim(E_1 + E_2) + \dim(E_1 \cap E_2) = \dim E_1 + \dim E_2 \)
PROOF:

Given $E_1$ and $E_2$ be two L-fuzzy vector subspaces then the sum of $E_1$ and $E_2$ be denoted by $E_1 + E_2$

$$(\dim(E_1 + E_2) + \dim (E_1 \cap E_2))^{(a)} = (\dim(E_1 + E_2))^{(a)} + (\dim(E_1 \cap E_2))^{(a)}$$

$$= \dim(E_1 + E_2)^{(a)} + \dim (E_1 \cap E_2)^{(a)}$$

$$= \dim(E_1)^{(a)} + \dim (E_2)^{(a)} + \dim (E_1 \cap E_2)^{(a)}$$

$$= \dim(E_1)^{(a)} + \dim (E_2)^{(a)}$$

Therefore $\dim(E_1 + E_2) + \dim (E_1 \cap E_2) = \dim E_1 + \dim E_2$

Hence the theorem.

CONCLUSION

In this paper L-fuzzy vector subspace is defined and showed that its dimension is an L-fuzzy natural number. Based on the definitions some properties of crisp vector space s are hold in finite dimensional vector spaces. In particular the equality $\dim(E_1 + E_2) + \dim (E_1 \cap E_2) = \dim E_1 + \dim E_2$ holds without any restricted conditions.

REFERENCE