Super (a,d)-C_n-antimagicness of Windmill Graphs $W(n,r)$

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Let $G=(V,E)$ be a finite simple graph with $|V(G)|$ vertices and $|E(G)|$ edges. An edge-covering of G is a family of subgraphs $H_1,H_2,\ldots,H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs H_i, $i=1,2,\ldots,t$. If every subgraph H_i is isomorphic to a given graph H, then the graph G admits an H-covering. A graph G admitting H covering is called an (a,d)-H-antimagic if there is a bijection $f:V\cup E\to\{1,2,\ldots,|V(G)|+|E(G)|\}$ such that for each subgraph H' of G isomorphic to H, the sum of labels of all the edges and vertices belonged to H' constitutes an arithmetic progression with the initial term a and the common difference d. For $f(V)=\{1,2,3,\ldots,|V(G)|\}$, the graph G is said to be super (a,d)-H-antimagic and for $d=0$ it is called H-supermagic.

In this paper, we investigate the existence of super (a,d)-C_n-antimagic labeling of Windmill graphs, Friendship graphs and subdivided friendship graphs for difference $d=0,1,\ldots,5$ and $n\geq 3$.

Keywords: Windmill graph $W(n,r)$, Friendship graph $W(3,r)$, Subdivided friendship graph $W(3,r)(m)$, super (a,d)-C_n-antimagic, super (a,d)-$C_{3(m+1)}$-antimagic.

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1 Introduction

An edge-covering of finite and simple graph G is a family of subgraphs $H_1,H_2,\ldots,H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs H_i, $i=1,2,\ldots,t$. In this case we say that G admits an $(H_1,H_2,\ldots,H_t)$-(edge) covering. If every subgraph H_i is isomorphic to a given graph H, then the graph G admits an H-covering. A graph G admitting an H-covering is called (a,d)-H-antimagic if there exists a total labeling $f:V(G)\cup E(G)\to\{1,2,\ldots,|V(G)|+|E(G)|\}$ such that for each subgraph H' of G isomorphic to H, the H-weights,

$$wt_f(H')=\sum_{v\in V(H')}f(v)+\sum_{e\in E(H')}f(e),$$

constitute an arithmetic progression $a,a+d,a+2d,\ldots,a+(t-1)d$, where $a>0$ and $d\geq 0$ are two integers and t is the number of all subgraphs of G isomorphic to H. Moreover, G is said to be super (a,d)-H-antimagic, if the smallest possible labels appear on the vertices. If G is a (super) (a,d)-H-antimagic graph then the corresponding total labeling f is called the (super) (a,d)-H-antimagic labeling. For $d=0$, the (super) (a,d)-H-antimagic graph is called (super) H-magic.

The (super) H-magic graph was first introduced by Gutiérrez and Lladó in [8]. They proved that the star $K_{1,n}$ and the complete bipartite graphs $K_{n,m}$ are $K_{1,h}$-supermagic for some h. They also proved that the path P_n and the cycle C_n are P_h-supermagic for some h. Lladó and Moragas [14] investigated C_n-(super)magic graphs and proved that wheels, windmills, books and prisms are C_h-magic for some h. Some results on C_n-supermagic labelings of several classes of graphs can be found in [18]. Maryati et al. [16] gave P_h-(super)magic labelings of shrubs, subdivision of shrubs and banana tree graphs. Other examples of H-supermagic graphs with different choices of H have been given by Jeyanthi and Selvagopal in [11]. Maryati et al. [17] investigated the G-supermagicness of a disjoint union of c copies of a graph G and showed that disjoint union of any paths is cP_h-supermagic for some c and h.

The (a,d)-H-antimagic labeling was introduced by Inayah et al. [9]. In [10] Inayah et al. investigated the super (a,d)-H-antimagic labelings for some shackles of a connected graph H.

For $H \cong K_2$, (super) (a,d)-H-antimagic labelings are also called (super) (a,d)-edge-antimagic total labelings. For further information on (super) edge-magic labelings, one can see [3, 4, 7, 15].

The (super) (a,d)-H-antimagic labeling is related to a (super) d-antimagic labeling of type $(1,1,0)$ of a plane graph which is the generalization of a face-magic labeling introduced by Lih [13]. Further information on super d-antimagic labelings can be found in [1, 2, 5].

In this paper, we study the existence of super (a,d)-C_n-antimagic labeling of Windmill graphs, Friendship graphs and subdivided friendship graphs for differences $d = 0,1,\ldots,5$ and $n \geq 3$.

2 Super (a,d)-C_n-antimagic labeling of Windmills

In this paper, we brought into account a family of super C_n-antimagic graphs for any integer $n \geq 3$ and for differences $d = 0,1,\ldots,5$.

Let C_n, $n \geq 3$ be a cycle of length n. The graph $W(n,r)$ obtained by identifying one vertex in each of $r \geq 2$ disjoint copies of the cycle C_n. The resulting graphs is called windmill, and $W(3,r)$ is also known as the friendship graph[14].

let v_n be the only common vertex of r copies of n-cycles in the windmill graph $W(n,r)$ and $H_1,\ldots,H_r$ be the n-cycles of $W(n,r)$. Clearly, we have $|V(W(n,r))| = |E(W(n,r))| = nr$.

Every n-cycle H_i in $W(n,r)$ has the vertex set $V(H_i) = \cup_{j=1}^{n-1}\{v_j^{(i)}\} \cup \{v_n\}$ and the edge set $E(H_i) = \cup_{j=1}^{n-1}\{v_j^{(i)} v_{j+1}^{(i)}\} \cup \{v_n v_1^{(i)}\}$.

Under a total labeling h, for the $H_i, i = 1,\ldots,r$, weight of $W(n,r)$ we have

$$wt_h(H_i) = \sum_{v \in V(H_i)} h(v) + \sum_{e \in E(H_i)} h(e).$$

$$= h(v_n) + \sum_{j=1}^{n-1} h(v_j^i) + \sum_{j=1}^{n} h(v_j^i v_{j+1}^i)$$

where indices are taken modulo n.

Theorem 1 For any two integers $r \geq 2$ and $n \geq 3$, the windmill $W(n,r)$ is super (a,d)-C_n-antimagic, where $d = 2k-1, k \in \mathbb{Z}^+$.

Proof. Consider the labeling h_p, $p = 2k-1, k \in \mathbb{Z}^+$ for vertices and edges of H_i, $i = 1,\ldots,r$ is defined as

$$h_p(v_j^{(i)}) = \begin{cases} 1 + r(j-1) + i, & j = 1,\ldots,n-1 \\ 1, & j = n \end{cases} \quad (1)$$

Clearly $h_p(V(H_i)) : i = 1,2,\ldots,r = 1,2,\ldots,r(n-1)+1\}$

The edges $E(H_i)$ are get labeled as

$$h_p(e_j^{(i)}) = \begin{cases} 1 + r(n+j-2) + i & \text{if } j = 1,\ldots,\lceil \frac{p}{2} \rceil \\ 2 + r(2n-j+1) - i & \text{if } j = \lceil \frac{p}{2} \rceil + 1,\ldots,n \end{cases} \quad (2)$$

Equations (??) and (??) show h_p is a total labeling and

$$wt_{h_p}(H_i) = n(2nr - r + 3) + r + m(mr - nr - 2r - 1) + i(2m - 1)$$

$$\{wt_{h_p}(H_i) = A + qi(2m-1), q = 1,\ldots,r\}$$

Where $A = n(2nr - r + 3) + r + m(mr - nr - 2r - 1)$ and $m = \lceil \frac{p}{2} \rceil$

From (??) it can be noted that for $d > 5$ odd, $n \geq \lceil \frac{p}{2} \rceil$.

This concludes the proof.

Theorem 2 The windmill $W(n,r)$ is super (a,d)-C_n-antimagic for any two integers $n \geq 3$ and $r > 1$ odd.

Proof. Consider the labeling h_p, $p = 0,2,4$ for vertices and edges of H_i, $i = 1,\ldots,r$ is defined as

$$h_p(x) = 1$$

$$h_p(v_1^{(i)}) = \begin{cases} \frac{i+1}{2} + 1 & \text{if } i = odd \\ \frac{r+1}{2} + \frac{i}{2} + 1 & \text{if } i = even \end{cases}$$

$$h_p(v_2^{(i)}) = \begin{cases} \frac{3r+1}{2} + \frac{i+1}{2} & \text{if } i = odd \\ \frac{i}{2} + (r+1) & \text{if } i = even \end{cases}$$

$$h_p(v_j^{(i)}) = 1 + r(j-1) + i \quad \text{if } j = 3,\ldots,n-1 \tag{3}$$

Clearly, the vertices get the smallest possible labels and

$$h_p(v_1^{(i)}) + h_p(v_2^{(i)}) = \frac{3r+5}{2} + i$$

The edges $E(G_i)$ are get labeled as

$$h_0(e_j^{(i)}) = \begin{cases} r(2n - j - 1) + 2 - i & \text{if } j = 1,\ldots,n-1 \\ 2r(n-1) + (1+i) & \text{if } j = n \end{cases}$$

$$h_2(e_j^{(i)}) = r(2n - j) + 2 - i \qquad \text{if } j = 1,\ldots,n-1$$

$$h_4(e_j^{(i)}) = \begin{cases} r(n + j - 2) + (1+i) & \text{if } j = 1,2,3 \\ 2 + r(2n - j + 3) - i & \text{if } j = 4,\ldots,n \end{cases}$$

The above assignment of vertices and edges of H_i is a total labeling and the weights are

$$wt_{h_p}(H_i) = \begin{cases} 2r(n^2-n+1)+3n-\dfrac{3r+1}{2}, & p=0 \\ 2nr(n-1)+\dfrac{6n+3r+1}{2}-2i, & p=2 \\ 2nr(n-1)+\dfrac{6n-3r-5}{2}+4i, & p=4 \end{cases}$$

which gives the super (a,d)-C_n-antimagic labeling of $W(n,r)$ for the differences $d=p, p=0,2,4$.

This concludes the proof.

3 Super (a,d)-C_n-antimagic labeling of Friendship Graphs

For $n=3$ windmill graph $W(3,r)$ is also known as the friendship graph. From theorem (1) and theorem(2), we can conclude that friendship graphs $W(3,r)$ and subdivided friendship graphs $W(3,r)(m)$ are super (a,d)-C_n-antimagic.

Corollary 1 The friendship graph $W(3,r)$, subdivided friendship graph $W(3,r)(m)$ are super (a,d)-C_3-antimagic and super (a,d)-$C_{3(m+1)}$-antimagic for any integers $n\geq 3, m,r\geq 1$ and for differences $d=1,3,\ldots,2k-1, k\in \mathbb{Z}^+$.

Corollary 2 The friendship graph $W(3,r)$, subdivided friendship graph $W(3,r)(m)$ are super (a,d)-C_3-antimagic and super (a,d)-$C_{3(m+1)}$-antimagic for any integers $n\geq 3, m\geq 1,\ r\geq 1$ odd and for differences $d=0,2,4$.

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