Observations on the ternary cubic Diophantine equation \(x^2 + y^2 - xy = 52z^3\)

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Abstract: The ternary cubic Diophantine equation is analyzed for its infinitely many non-zero distinct integral solutions. A few interesting properties among the solutions are presented.

Keywords: Ternary cubic equation, integral solutions

1. Introduction

Integral solutions for the cubic homogeneous or non-homogeneous diophantine equations is an interesting concept, as it can be seen from [1-3]. In [4-13] a few special cases of ternary cubic diophantine equations are studied. In this communication, we present the integral solutions of yet another ternary cubic equation \(x^2 + y^2 - xy = 52z^3\). A few interesting relations between the solutions are obtained.

Notations

- \(P_n^m\) = Pyramidal number of rank \(n\) with size \(m\)
- \(SO_n\) = Stella Octangular number of rank \(n\)
- \(Ob_l\) = Oblong number of rank \(l\)
- \(OH_n\) = Octahedral number of rank \(n\)
- \(Th_n\) = Thabit-ibn-Kurrah number
- \(C_{aro}\) = Carol number
- \(M_n\) = Mersenne number
- \(Ky_n\) = Kynea number
- \(t_{m,n}\) = Polygonal number of rank \(n\) with size \(m\)

2. Method of Analysis

The cubic equation under consideration is

\[x^2 + y^2 - xy = 52z^3 \quad (1)\]

Assuming \(x = u + v, y = u - v, u \neq v\) \( (2)\) in (1), it is written as

\[u^2 + 3v^2 = 52z^3 \quad (3)\]

Here, we rewrite 52 in five different ways and hence obtain different patterns of solutions to (1) which are illustrate as follows:

**Pattern 1**

Assume \(z = z(a, b) = a^2 + 3b^2, a, b \neq 0\)

Write 52 as

\[52 = (7+i\sqrt{3})(7-i\sqrt{3}) \quad (5)\]

Using (4) and (5) in (3) and employing the method of factorization, define

\[(u+i\sqrt{3}v) = (a+i\sqrt{3}b)^3 (7+i\sqrt{3}) \quad (6)\]

On comparing real and imaginary parts on both sides, we get

\[u = u(a, b) = 7a^3 - 9a^2b - 63ab^2 + 9b^3\]
\[v = v(a, b) = a^3 + 21a^2b - 9ab^2 - 21b^3\]

Substituting the values of \(u\) and \(v\) in (2), we get...
\[x = x(a, b) = 8a^3 + 12a^2b - 72ab^2 - 12b^3\]  \hspace{1cm} (7)
\[y = y(a, b) = 6a^3 - 30a^2b - 54ab^2 + 30b^3\]  \hspace{1cm} (8)

Thus (4), (7) and (8) represent the non-zero distinct integer solutions of (1)

**Properties**

1) \[x(a, 1) - y(a, 1) - 4P_a + t_{102a} + t_{32a} \equiv -20 \mod (22)\]
2) \[y(a, 1) + z(a, 1) - 4SO_a + t_{138a} + t_{12a} \equiv -9 \mod (55)\]
3) \[y(1,b) - z(1,b) - 450Oh_a + t_{52a} + t_{66a} \equiv 5 \mod (100)\]
4) \[\exists (a, a+1) + x(a, a+1) - 24P_a + 72P_a + 2SO_a + t_{30a} + t_{32a} \equiv -51 \mod (114)\]
5) \[z(a, a + 1) - t_{58a} + t_{50a} \equiv 3 \mod (10)\]

**Pattern 2**

Instead of (5), we write 52 as
\[52 = (5 + i\sqrt{3})(5 - i\sqrt{3})\]  \hspace{1cm} (9)

Following the procedure presented in pattern 1, we obtain the integer value of \(x, y, z\) satisfy (1) to be
\[x = x(a, b) = 8a^3 - 12a^2b - 72ab^2 + 12b^3\]  \hspace{1cm} (10)
\[y = y(a, b) = 2a^3 - 42a^2b - 18ab^2 + 42b^3\]  \hspace{1cm} (11)
\[z = z(a, b) = a^3 + 3b^2\]

Thus (4), (10) and (11) represent the non-zero distinct integer solutions of (1)

**Properties**

1) \[x(2a) + y(2a) - 3SO_a - t_{138a} - t_{118a} \equiv -87 \mod (153)\]
2) \[x(1, b) - 6SO_a + t_{166} \equiv 8 \mod (77)\]
3) \[z(2, b) = \{y(2a) - 6SO_a + t_{166} \equiv 20 \mod (220)\]
4) \[z(a, a + 1) - 3SO_a - t_{32a} + t_{14a} \equiv 0 \mod (12)\]
5) \[z(2^n, 2) - 13 \equiv M_{2n}\]

**Pattern 3**

52 can also be written as
\[52 = (2 + i\sqrt{3})(2 - i\sqrt{3})\]  \hspace{1cm} (12)

Proceeding as in pattern 1, the non-zero distinct integral solutions of (1) are given by
\[x = x(a, b) = 2a^3 - 30a^2b - 54ab^2 + 30b^3\]  \hspace{1cm} (13)
\[y = y(a, b) = -2a^3 - 42a^2b + 18ab^2 + 42b^3\]  \hspace{1cm} (14)

Thus (4), (13) and (14) represent the non-zero distinct integer solutions of (1)

**Properties**

1) \[z(2, b) + y(2, b) - 7SO_a - t_{12} + t_{106} \equiv -12 \mod (155)\]
2) \[y(1, b) - z(1, b) - 21SO_a + t_{22} + t_{42} \equiv -3 \mod (6)\]
3) \[x(a, 1) - 3SO_a + t_{102a} - t_{42a} \equiv 30 \mod (81)\]
4) \[z(1, b + 1) - t_{30b} + t_{84b} \equiv 4 \mod (9)\]
5) \[z(2^n + 1, 1) - 5 \equiv k_b\]

**Pattern 4**

Apart from (5), (9) and (12) we write 52 as
\[52 = \frac{[20 + i\sqrt{3}](20 - i\sqrt{3})}{49}\]  \hspace{1cm} (15)

For this choice, the corresponding solutions of (1) are represented by
\[x = x(A, B) = 2646A^3 + 5586A^2B - 23814AB^2 - 5586B^3\]  \hspace{1cm} (16)
\[y = y(A, B) = 2254A^3 + 9114A^2B - 20286AB^2 + 9114B^3\]  \hspace{1cm} (17)
\[z = z(A, B) = 49A^2 + 147B^2\]  \hspace{1cm} (18)

**Properties**

1) \[y(1, n) - 13671OH_a + t_{10002} + t_{574, 8} \equiv 2254 \mod (3955)\]
2) \[y(1, b) - x(1, b) - 735SO_a - t_{902b} + t_{196b} \equiv 392 \mod (3822)\]
3) \[z(1, b) - x(1, b) - 8379OH_a - t_{49002} + t_{10800, 8} \equiv -2597 \mod (15582)\]
4) \[z(2^n - 1, 1) - 245 = 49 \text{ Carol}_n\]
5) \[z(1, 2^n) - 98 = 49 \text{ Tha}_{2n}\]
Pattern 5

Equation (3) can be written as
\[ u^2 + 3v^2 = 32z^3 + 1 \]  
(19)

Write ‘1’ as
\[ 1 = (1+i\sqrt{3}) (1 - i\sqrt{3}) \]  
(20)

Define \( (u+i\sqrt{3}v) = \frac{(7+i\sqrt{3})(a+i\sqrt{3}b)}{2} (1+i\sqrt{3}) \)  
(21)

Equating real and imaginary parts, we have
\[ u = 2(a^2 - 18ab^2 - 9ab^2 + 18b^3) \]
\[ v = 2(2a^2 + 3a^2b - 18ab^2 - 3b^3) \]

Since our aim is to find integer solution substituting the values of \( u,v \) in (2), we obtain the distinct nonzero integral solutions to (1) as
\[ x = x(a,b) = 2(3a^2 - 15a^2b - 27ab^2 + 15b^3) \]
\[ y = y(a,b) = 2(-a^2 - 21a^2b + 9ab^2 + 21b^3) \]

Properties
1. \( x(1,b) + y(1,b) - 36SO_4 + 42SO_4 - 10b \equiv -4 \text{mod} 72 \)
2. \( x(a,a+1) + 3SO_4 + 6OP_5 + 112a^2 + 10b_8 \equiv 7 \text{mod} 8 \)
3. \( z(a+1,a) - t \text{mod 170}a + 170a \equiv 1 \text{mod 6} \)
4. \( y(2^3,1) - 16 = 18M_{2a} - 42M_{2a} - 2M_{3n} \)

Conclusion
To conclude one may search for other patterns of solutions to (1) along with their properties.

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