Domination number of some graphs

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***Abstract*: A set D of vertices in a graph G is a *dominating set* if every vertex in $V-D$ is adjacent to some vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality of the dominating set of G. In this paper, we discuss the dominating set and domination number of the graphs such as fan $F_{m,2}$, diamond snake D_n, banana tree $B(m,n)$, coconut tree $CT(m,n)$, firecracker $F(m,n)$.**

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***Keywords*: Dominating set; Domination number; fan graph; diamond snake; banana tree; coconut tree; firecracker graph.**

1. INTRODUCTION

All the graphs considered here are finite, undirected with no loops and multiple edges. A *Graph* G consists of a pair (V,E) where V is a non-empty finite set whose elements are called *vertices or nodes* and E is a set of unordered pairs of distinct elements of V. The elements of E are called *edges* of the graph G. The *degree* of a vertex of a graph is the number of edges incident to the vertex, with loops counted twice. The degree of a vertex v is denoted by $\deg(v)$. The maximum and minimum degree of a graph is denoted by $\Delta(G)$ and $\delta(G)$ respectively. $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v respectively. A vertex $v \in G$ is called *pendant vertex* or end vertex of G if $\deg(v)=1$. An edge of a graph is said to be *pendant* if one of its vertices is a pendant vertex.
For graph theoretic terminology we refer to Chartrand and Lesniak [4].

The rigorous study of dominating sets in Graph theory began around 1960. According to Hedetneimi and Laskar (1990) [8], the domination problems were studied from the 1950's onwards, but the rate of research on domination significantly increased in the mid 1970's. In 1958, Berge [1] defined the concept of the domination number of a graph, called as "coefficient of external stability". In 1962, Ore [15] used the name "dominating set" and "domination number" for the same concept. In 1977 Cockayane and Hedetneimi [5] made an interesting and extensive survey of the results known at the time about dominating sets in graphs[7], [14]. They have used the notation $\gamma(G)$ for the domination number of a graph, which has become very popular since then [3]. The survey paper of Cockayane and Hedetniemi [5] has generated lot of interest in the study of dominating sets in graphs. Recent books on domination, have stimulated a sufficient inspiration leading to the expansive growth of this field to study. It has also put some order into this huge collection of research papers, and organized the study of dominating sets in graphs into meaningful subareas, placing the study of dominating sets in even broader mathematical and algorithmic contexts [6], [7] and [17]. Domination can be useful tool in many chemical structures [16] also there is many applications of domination theory in wireless communication networks [9] business network and making decisions.

The domination number for the helm graph H_n and web graph W_n were proved by Ayhan A. Khalil [11]. The domination number for the friendship graph F_n and windmill graph $Wd(m,n)$ were proved by Dr. C S Nagabhushana et al. [13]. The domination number for the tadpole graph $T_{m,n}$ has proved by Murthy K. B[12]. The domination number for the book graph B_n and stacked book graph $B_{3,n}$ were proved by Kavitha B N and Indrani Kelkar[10]. In this paper we discuss the dominating set and domination number of the graphs such as fan $F_{m,2}$, diamond snake D_n, banana tree $B(m,n)$, coconut tree $CT(m,n)$, firecracker $F(m,n)$.

2. DEFINITIONS AND NOTATIONS

Definition 2.1: A set D of vertices in a graph $G=(V,E)$ is called a *dominating set* of G, if every vertex in $V-D$ is adjacent to some vertex in D. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of the dominating set in G [2], [18].

Definition 2.2: A *fan* graph $F_{m,n} = \overline{K_m} + P_n$, where $\overline{K_m}$ is the empty graph (consists of m isolated nodes with no edges) and P_n is the path graph on n nodes.

Definition 2.3: An *fire cracker* $F(m,n)$ is a graph obtained by the series of interconnected m copies of n stars by linking one leaf from each.

Definition 2.4: The graph G consists of collection of n cycles C_4, these cycles are connected in such a way that any two adjacent cycles sharing a common vertex, the resulting graph is called the *diamond snake* graph and it is denoted by D_n. A diamond snake has $3n+1$ vertices and $4n$ edges, where n is the number of blocks in the diamond snake. A snake is an Eulerian path that has no chords.

Definition 2.5: A *Banana tree* $B(m,n)$ is a graph obtained by connecting one leaf of each of m copies of a n - star graph with a new single root vertex $'v'$. Note that the edges contain pendant nodes are called the tree leaves.

Definition 2.6: [22] A *Coconut tree* $CT(m,n)$ is the graph obtained from the path P_m by appending $'n'$ new pendant edges at an end vertex of P_m.

Definition 2.7: The *floor function* of a real number x is the greatest integer less than or equal to x and it is denoted by $\lfloor x \rfloor$. Suppose that $n \le x < n+1$, where n is an integer, then $\lfloor x \rfloor = n$.

Definition 2.8: The *ceiling function* of a real number x is the lowest integer greater than or equal to x and it is denoted by $\lceil x \rceil$. Suppose that $n-1 < x \le n$, where n is an integer, then $\lceil x \rceil = n$.

3. DOMINATION NUMBER

Theorem 3.1. The domination number of any fan graph $F_{m,2}$ is 1, where $m \ge 1$.

Proof. Let $G \cong F_{m,2}$ be a fan graph on $m+2$ vertices with $2m+1$ edges and let D be a minimum dominating set of graph G. By definition of the fan graph, the graph $G = \overline{K_m} + P_2$ where $\overline{K_m}$ is the empty graph on m nodes; P_2 is the path graph on 2 nodes. Let $V(P_2) = \{u, v\}$.

There are 2 nodes are available in path P_2 of fan graph, so that if we choose any one vertex from the path P_2, then all the other vertices of G are dominated by our chosen vertex. So we will get a minimum dominating set and its cardinality is the domination number of graph G.

Hence the dominating set D of $G = \{u\}$ or $\{v\}$.

Therefore, the domination number of graph G is 1.

That is, $\gamma(G) = 1$.

Example

The dominating set and domination number of fan graphs $F_{3,2}$ and $F_{4,2}$ are shown in Figure 3.1 and Figure 3.2 respectively.

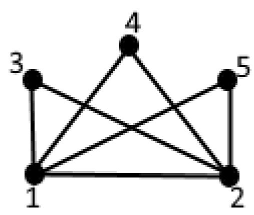


Fig. 3.1. Fan $F_{3,2}$, $D = \{1\}, \gamma(G) = 1$

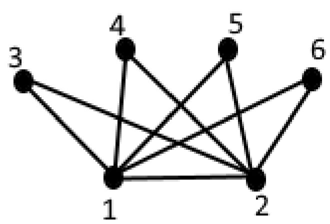


Fig. 3.2. Fan $F_{4,2}$, $D = \{2\}, \gamma(G) = 1$

Theorem 3.2. For any firecracker graph $F(m,n)$, the domination number is m, where $n \geq 2$.

Proof. Let $G \cong F(m,n)$ be a firecracker graph on mn vertices with $(mn)-1$ edges and let D be a minimum dominating set of graph G. By definition of the firecracker graph, the graph is obtained from series of interconnected m copies of n stars by linking one leaf from each.

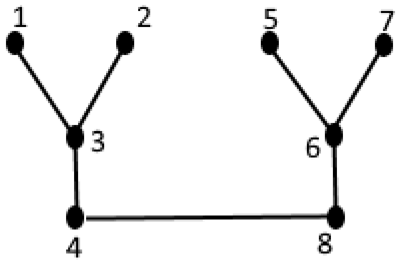
For each of the n stars, if we choose all of the central vertices as one set, it will be dominate all the other vertices of G. So we will get a minimum dominating set and its cardinality is the domination number of graph G.

Therefore, the domination number of G is m.

That is, $\gamma(G) = m$.

Example

The dominating set and domination number of firecracker graphs $F(2,4)$ and $F(3,4)$ are shown in Figure 3.3 and Figure 3.4 respectively.



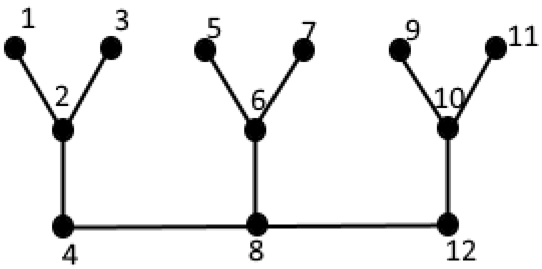


Fig. 3.3. Firecracker graph F(2,4)
Fig. 3.4. Firecracker graph F(3,4)

$D = \{3,6\}, \gamma(G) = 2$ $\quad$ $D = \{2,6,10\}, \gamma(G) = 3$

Theorem 3.3. For any diamond snake graph D_n, the domination number is $n+1$, where $n \geq 1$.

Proof. Let $G \cong D_n$ be a diamond snake graph on $3n+1$ vertices with $4n$ edges and let D be a minimum dominating set of graph G. By definition of the diamond snake graph, the graph G consists of collection of n cycles C_4, these cycles are connected in such a way that any two adjacent cycles sharing a common vertex, where n is the number of blocks in the diamond snake.

If we choose any one of the vertices of degree 2 from the first and last copies of G, then we choose all common vertices which are sharing by consecutive cycles of G. So we will get a minimum dominating set and its cardinality is the domination number of graph G. Therefore, the domination number of G is $n+1$. That is $\gamma(G) = n+1$.

Example

The dominating set and domination number of diamond snake graph D_2 is shown in Figure 3.5.

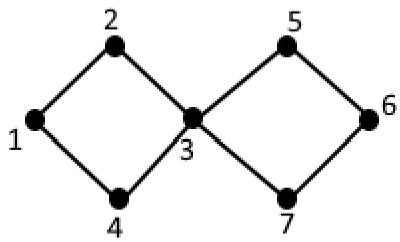


Fig. 3.5. Diamond snake graph D_2

$D = \{1,3,6\}, \gamma(G) = 3$

Theorem 3.4. For any banana tree $B(m,n)$, the domination number is $\gamma(G) = m+1$, where $m \geq 1, n \geq 3$.

Proof. Let $G \cong B(m,n)$ be a banana tree on $(mn)+1$ vertices with mn edges and let D be a minimum dominating set of graph G. By definition of the banana tree, the graph is obtained by connecting one leaf of each of m copies of a n-star graph with a new single root vertex $'v'$.

We distinguish three cases to obtain the domination number of graph G.

Case 1. Let $n = 1$.

We have to find the domination number of banana tree graph $B(m,1)$ which is shown in Fig. 3.6.

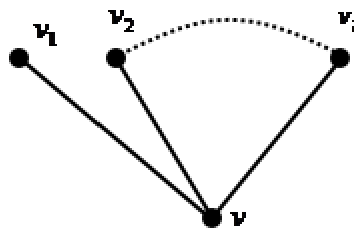


Fig. 3.6. Banana tree $B(m,1)$ graph

Then $D = \{v\}$ is the minimum dominating set of graph G.

Therefore, $\gamma(G) = 1$.

Case 2. Let $n = 2$

We have to find the domination number of banana tree graph $B(m,2)$ which is shown in Fig. 3.7.

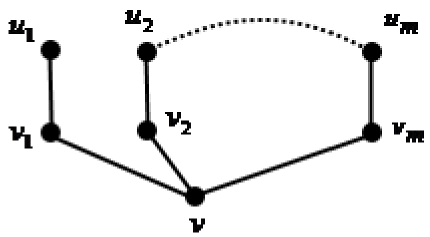


Fig. 3.7. Banana tree $B(m,2)$ graph

Then $D = \{v_1, v_2, ..., v_m\}$ is the minimum dominating set of graph G.

Therefore, $\gamma(G) = m$.

Case 3. Let $G \cong B(m,n)$, where $n \geq 3$.

Moreover $\deg(v) = \Delta(G)$. So $'v'$ must be included in an any minimum dominating set of G. For G, if we choose the apex vertices of every star graph it will dominate all the other vertices except single root vertex. Hence $\gamma(G) = m + |\{v\}| = m+1$.

Therefore, the domination number of G is $m+1$.

Hence $\gamma(G) = m+1$.

Example

The dominating set and domination number of banana tree graphs $B(2,5)$ and $B(3,4)$ are shown in Figure 3.8 and Figure 3.9 respectively.

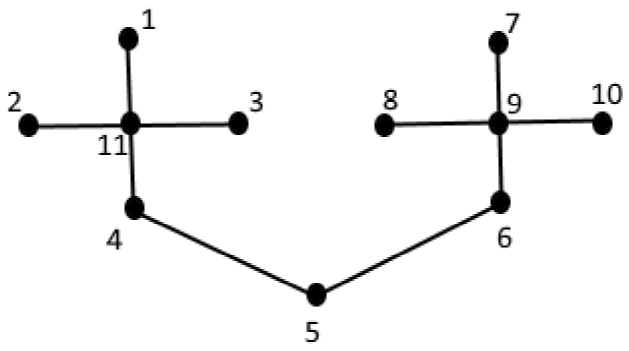


Fig. 3.8. Banana tree $B(2,5)$

$D = \{11,9,5\}, \gamma(G) = 3$

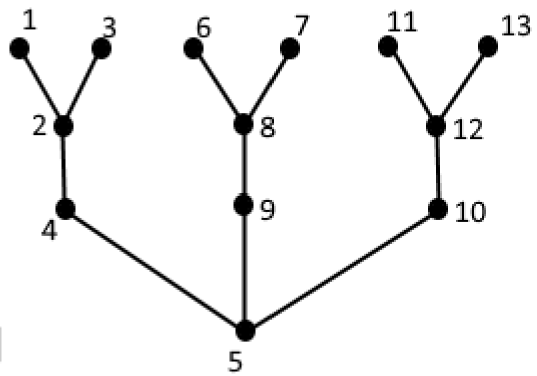


Fig. 3.9. Banana tree $B(3,4)$

$D = \{2,8,12,5\}, \gamma(G) = 4$

Theorem 3.5. For any coconut tree $CT(m,n)$, the domination number is $1+\left\lceil \frac{m-2}{3} \right\rceil$, where $m \geq 1, n \geq 1$.

Proof. Let $G \cong CT(m,n)$ be a coconut tree on $m+n$ vertices with $m+(n-1)$ edges and let D be a minimum dominating set of graph G. By definition of coconut tree, the graph is obtained from the path P_m by appending n new pendant edges at an end vertex v_1 (say) of P_m.

Clearly the vertex v_1 must be included in any minimum dominating set, since $\deg(v_1) = \Delta(CT(m,n))$. In G, v_1 dominate all pendant vertices attached with v_1.

Therefore, the domination number of G is $1+\gamma(P_{m-2}) = 1+\left\lceil \frac{m-2}{3} \right\rceil$.

Hence $\gamma(G) = 1+\left\lceil \frac{m-2}{3} \right\rceil$.

Example

The dominating set and domination number of coconut tree graph $CT(4,7)$ is shown in Figure 3.10.

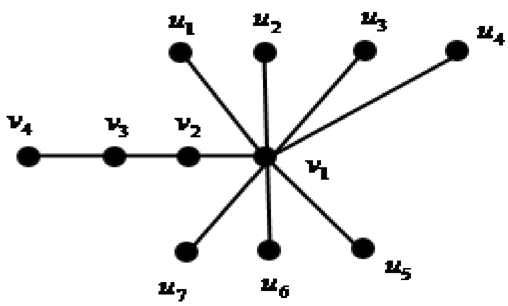


Fig. 3.10. Coconut tree $CT(4,7)$

The dominating set of $CT(4,7)$ is $D = \{v_1, v_4\}$ and $\gamma(G) = 2$.

4. CONCLUDING REMARKS

The theory of domination plays vital role and many researchers are producing a huge collection of papers in this topic. The domination concept is further classified into equitable dominating and end equitable dominating sets. In this paper we proved that the dominating set and domination number of the graphs such as fan $F_{m,2}$, diamond snake D_n, banana tree $B(m,n)$, coconut tree $CT(m,n)$, firecracker $F(m,n)$.

REFERENCES

[1] C. Berge, Theory of graphs and its applications, Dunod, Paris, (1958).
[2] C. Berge, Graphs and hypergraphs, North-Holland, Amsterdam, (1973).
[3] B. Bollob'as and E. J. Cockayne , Graph-theoretic parameters concerning domination independence and irredundance, *J. Graph theory*, 3(1979), 241-249.
[4] G. Chartrand, Lesniak, Graphs and Digraphs, fourth ed., CRC Press, Boca Raton, 2005.
[5] E. Cockayne and S. Hedetniemi, Towards a theory of domination in graphs, Networks Fall (1977), 247-261.
[6] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, "Domination in Graphs-Advanced Topics", New York: Dekker, 1998.
[7] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, "Fundamentals of domination in Graphs", New York: Dekker, 1998.
[8] S. T. Hedetniemi and R. C. Laskar, "Topics on domination", North Holland 1991.
[9] J. L. Hurink and T. Nieberg, "Approximating minimum independent dominating sets in wireless networks", Information processing letters, 109(2008), 155-160.
[10] B. N. Kavitha and Indrani Kelkar, "Split and Equitable domination in book graph and stacked book graph", *International Journal of Advanced Research in Computer Science*, 8(6)2017.
[11] Ayhan A. Khalil, "Determination and testing the Domination Numbers of Helm Graph, Web graph and Levi Graph Using MATLAB", J. Edu. & Sci., 24(2)2011.
[12] K. B. Murthy, "The End equitable Domination of Dragon and some Related Graphs", *Journal of Computer and Mathematical sciences*, 7(3)(2016), 160-167.
[13] Dr. C. S. Nagabhushana, B. N. Kavitha and H. M. Chudamani, "Split and Equitable Domination of Some Special Graph", *International Journal of Science Technology & Engineering*, 4(2)2017.
[14] U. Niranjan, R. B. V. Subramanyam, V. Khanaa., "Developing a Web Recommendation System Based on Closed Sequential Patterns", *Communications in Computer and Information Science*, 1010 (2010), 171-179.
[15] O. Ore, Theory of Graphs, Amer. Math. Soc. Colloquium Pub., *Amer. Math. Soc., Providence*. Rhode Island, 38(1962), 206.
[16] D. Vuki and A. Klobu , "K-dominating sets on linear benzenoids and on the infinite hexagonal grid", *Croatica Chemica Acta*., 80(2)(2007), 187-191.
[17] H. B. Walikar, B. D. Acharya and E. Sampath kumar, Recent developments in the theory of domination in graphs, *Mehta Research institute*, Allahabad, MRI Lecture notes in Math. 1 (1979).
[18] B. Zmazek and J. Zerovnik , "On domination numbers of graph bundles", *Institute of mathematics, physics and mechanics*, 43(2005), 1-10.