Bipolar-valued fuzzy BZMV algebra

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Abstract: Brower-Zadeh MV – algebra is the result of a natural pasting between Brower – Zadeh algebras and MV – algebras. In this study the Bipolar – valued fuzzy values are introduced to Brower-Zadeh MV – algebras and the Strong s, t cuts are also defined.

Index Terms - BZMV- algebra, BZMV $\alpha\delta$- algebra, Bipolar – valued fuzzy, strong cut

Introduction:

MV algebra have been introduced by C, CHANG [2]in order to provide an adequate semantic characterisation for Lukasiewicz many valued logics. (i.e)complete with respect to the evaluations of propositional variables in the real unit interval[0,1].

Recall in fact that the prototypical example of an MV-algebra is the standard one [0,1]MV = < [0, 1], $\oplus$, $\neg$, 0 > where for all x,y $\in$ [0,1],

\[
\mu(x \oplus y) = \min \{1, x + y\},
\]

\[
\mu(x \neg y) = \neg(\mu(\neg x \oplus \neg y) \oplus x),
\]

\[
x \neg = x
\]

\[
x \neg \neg x = \neg x = 1 - x.
\]

In 1965, Zadeh introduced the notion of a fuzzy subset of a set. Since then it has become a vigorous area of research in different domains. There have been a number of generalizations of this fundamental concepts such as intuitionistic fuzzy sets, interval-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0,1] to [-1,1]. In a bipolar valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree (0,1] indicates that elements somewhat satisfy the property, and the membership degree [-1,0) indicates that elements somewhat satisfy the implicit counter-property. Bipolar-valued fuzzy sets and intuitionistic fuzzy sets look similar to each other. However, they are different from each other. Thus the bipolar-valued fuzzy set concepts are applied to the BZMV ALGEBRAS and some of the properties are verified.

Preliminaries:

[2] Definition 1.1 :

A Brower Zadeh MV algebra (shortly BZMV algebra) is a structure $A = < A, \oplus, \neg, \sim, 0 >$, where $A$ is a non empty set of elements,0 is a constant element of $A$, $\neg$ and $\sim$ are unary operations on $A$, $\oplus$ is a binary operation on $A$. The following axioms hold:

\* $(x \oplus y) \oplus z = (y \oplus z) \oplus x \oplus x \oplus 0 = x$

\* $\neg(\neg x) = x$

\* $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$

\* $x \sim \sim x = 0$

\* $x \neg \neg x = \neg x$

\* $\neg(\neg x \oplus y) \oplus y = \neg(\sim x \oplus \sim y) \oplus \sim y$

[17] Definition 1.2:

A de- Morgan Brower Zadeh MV algebra (shortly BZMV dM algebra) is a structure $A = < A, \oplus, \neg, \sim, 0 >$ that satisfies the axioms

\* $(x \oplus y) \oplus z = (y \oplus z) \oplus x$

\* $x \oplus 0 = x$

\* $\neg(\neg x) = x$

\* $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$

\* $\neg x \sim \sim x = 0$

\* $x \sim \sim x = \sim x$

\* $\neg(\neg x \oplus y) \oplus y = \neg(\sim x \oplus \sim y) \oplus \sim y$

and also the following condition:

\* $\neg(\neg x \oplus \neg y) \oplus \neg y = \neg(\sim x \oplus \sim \sim y) \oplus \sim \sim y$

Bipolar – valued fuzzy BZMV algebra:

Definition 2.1:[29]

Let $G$ be a non empty set. A Bipolar-valued fuzzy set in $G$ is an object having the form

$B = \{ (x, \mu^+(x), \mu^-(x)) ; x \in G \}$ where $\mu^+: G \rightarrow [0, 1]$ and $\mu^-: G \rightarrow [-1; 0]$ are mapping.

Note: In this paper we use the symbol $B = (\mu^+, \mu^-)$ for the Bipolar-valued fuzzy set

$B = \{ (x, \mu^+(x), \mu^-(x)) ; x \in G \}$
Definition 2.2:
A Bipolar-valued fuzzy set \( B = (\mu^+, \mu^-) \) is called a "Bipolar-valued fuzzy BZMV -algebra" (BFBZMV) of \( M \), if for every \( x, y \) in \( M \) it satisfies:
i) \( \mu^+(x) \leq \mu^+(\neg x) \)
ii) \( \mu^- (x) \geq \mu^- (\neg x) \)
iii) \( \inf \mu^+ (x) \geq \min \{\mu^+ (x), \mu^+ (y)\} \)
iv) \( \sup \mu^- (x) \leq \max \{\mu^- (x), \mu^- (y)\} \)

Definition 2.3:
Let \( M \) be a nonempty set endowed with an operation \( \oplus \), an unary operation \( \sim \) and \( \neg \) and a constant 0 satisfying the following axioms,
for all \( x, y, z \in M \):
• \( (x \oplus y) \oplus z = (y \oplus z) \oplus x \)
• \( x \oplus 0 = x \)
• \( \neg(\neg x) = x \)
• \( \neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x \)
• \( x \oplus \sim x = \neg x \)
• \( x \oplus \sim \sim x = \sim x \)
• \( \neg \sim [\neg(\neg x \oplus y) \oplus y] = \neg(\sim x \oplus \sim y) \oplus \sim y \)

For every subsets \( A \) and \( B \) of \( M \) we define the operators as follows:
\[
a \oplus b = \begin{cases} a + b, & \text{if } a + b < 1 \text{ and also if } a, b \in \mu^+ \\ 1, & \text{otherwise} \end{cases}
\]
\[
\neg a = \begin{cases} -a, & \text{if } a \in \mu^+ \\ -1, & \text{if } a \in \mu^- \\ 1, & \text{if } a = 0 \text{ and also } a \in \mu^+ \\
0, & \text{otherwise} \end{cases}
\]
\[
\sim a = \begin{cases} -1, & \text{if } a = 0 \text{ and also } a \in \mu^- \\ 0, & \text{otherwise} \end{cases}
\]

EXAMPLE 2.1:
Let \( M = \{0, a, 1\} \) and define \( \oplus, \neg, \sim \) by the following tables:
\[
a \oplus b = \begin{cases} a + b, & \text{if } a + b < 1 \text{ and also if } a, b \in \mu^+ \\ 1, & \text{otherwise} \end{cases}
\]
\[
\neg a = \begin{cases} 1, & \text{if } a \in \mu^+ \\ -1, & \text{if } a \in \mu^- \\ 0, & \text{otherwise} \end{cases}
\]
\[
\sim a = \begin{cases} 0, & \text{if } a = 0 \text{ and also } a \in \mu^+ \\ 1, & \text{otherwise} \end{cases}
\]

Now define \( \mu^+ \) and \( \mu^- \) as follows:
\[
\mu^+(x) = \begin{cases} 0.7, & \text{if } x \in \mu^+ \\ 0.2, & \text{otherwise} \end{cases}
\]

\[
\mu^-(x) = \begin{cases} 0.7, & \text{if } x \in \mu^- \\ 0, & \text{otherwise} \end{cases}
\]
If \( a \in \mu^- \)

<table>
<thead>
<tr>
<th>X</th>
<th>0</th>
<th>a</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu^-(x) )</td>
<td>-1</td>
<td>-0.3</td>
<td>-1</td>
</tr>
</tbody>
</table>

Then \( M = \{(0,a,1), \sim, 0 \} \) is a Bipolar valued fuzzy BZMV algebra (BFBZMV) of \( M \).

**Proposition 2.1:**
Let \( \mu^+; \mu^- \) be a BZMV algebra of \( M \).
Then \( B = (\mu^+; \mu^-) \) is a Bipolar-valued fuzzy BZMV algebra of \( M \). Conversely, if \( B = (\mu^+; \mu^-) \) is a Bipolar-valued fuzzy BZMV algebra of \( M \), then \( (\mu^+; \mu^-) \) are BZMV algebra of \( M \).

**Proof:**
Let \( x, y \in M \), we know that,
\[
\min\{-\mu(x), -\mu(y)\} = -\max\{\mu(x), \mu(y)\} = \min[\mu_B^+(x), \mu_B^-(y)]
\]
and also,
\[
sup_{x \in \mu(y)}(\mu_B^+(x)) = \max\{sup_{x \in \mu(y)}(\mu_B^+(x)), sup_{x \in \mu(y)}(\mu_B^-(y))\} \leq \max\{\max[\mu_B^+(x), \mu_B^-(y)], \max[\mu_B^-(x), \mu_B^+(y)]\} = \max\{\mu_B^+(x), \mu_B^-(x)\}
\]
Hence Proved.

**Definition 2.4 :**
Let \( A = \langle A, \oplus, \neg, \sim, 0 \rangle \) be BZMV algebra and \( S \subseteq A \) be a non empty set containing "0". If \( S \) is a sub-structure BZMV algebra of \( A \) with respect to "\( \oplus \)" and "\( \sim \)" then we say that \( S \) is a MV algebra of \( A \).

**Lemma 2.1:**
Let \( A = \langle A, \oplus, \neg, \sim, 0 \rangle \) be BZMV algebra and \( S \subseteq A \) be a non empty set containing "0". Then \( S \) is a MV algebra of \( A \) iff for all \( x, y \in S \):
* \( x \oplus y \in S \)
* \( \sim x \in S \)

**Definition 2.5 :**
Let \( A = \langle A, \oplus, \neg, \sim, 0 \rangle \) be BZMV algebra and \( S \subseteq A \) be a non empty set containing "0". If \( S \) is a sub-structure BZMV algebra of \( A \) with respect to "\( \oplus \)" and "\( \sim \)" then we say that \( S \) is a MV algebra of \( A \).

**Lemma 2.2:**
Let \( A = \langle A, \oplus, \neg, \sim, 0 \rangle \) be BZMV algebra and \( S \subseteq A \) be a non empty set containing "0". Then \( S \) is a MV algebra of \( A \) iff for all \( x, y \in S \):
* \( x \oplus y \in S \)
* \( \sim x \in S \)

**Definition 2.6:**
A Bipolar-valued fuzzy set \( B = (\mu^+; \mu^-) \) is called a "Bipolar-valued fuzzy MV - algebra" (BFMV) of \( A \), if for every \( x, y \in A \) it satisfies:

i) \( \mu^+(x) \leq \mu^+(\sim x) \)

ii) \( \mu^-(x) \geq \mu^-((\sim x)) \)

iii) \( \inf \mu^+(x) \geq \min\{\mu^+(x), \mu^+(y)\} \)

iv) \( \sup \mu^-(x) \leq \max\{\mu^-(x), \mu^-(y)\} \)

**Remark 2.1 :**
By the definition \( \sim \sim (\sim x) = x \) we have:
\[
\mu^+(\sim x) \geq \mu^+(\sim x) \Rightarrow \mu^+(x) \leq \mu^+(\sim x)
\]
\[
\mu^-(\sim x) \leq \mu^-((\sim x)) \Rightarrow \mu^-(\sim x) \leq \mu^-(x)
\]
Hence conditions (i),(ii) in definition can be written as:

(i) \( \mu^+(x) = \mu^+(\sim x) \)

(ii) \( \mu^-(x) = \mu^-(\sim x) \)

**Strong positive t-cut and Strong negative s - cut**

**Definition 3.1:**
Let \( B = (\mu^+; \mu^-) \) be a Bipolar-valued fuzzy set of \( A \) and \((s,t) \in [-1,0) \times [0,1] \).

Then:
* The set \( B_s^+ = \{x \in A; \mu^+(x) \geq t\} \) is called positive t-cut of \( B \).
* The set \( B_s^- = \{x \in A; \mu^- (x) \leq s\} \) is called negative s-cut of \( B \).
The set \( B^+_x \) is called strong positive s-cut of \( B \)

The set \( B^-_x \) is called strong negative s-cut of \( B \)

The set \( A^+_{y\mid t} \) is called \((t,s)\) cut of \( B \)

The set \( A^-_{y\mid t} \) is called \((t,s)\) cut of \( B \)

Now let \( a \in A \),

Then

\[
\inf_{x \in A} \mu^+(y) = \min \{ \mu^+(x), \mu^+(y) \}
\]

Hence \( \mu^+(a) \geq \mu^+(0) \) \( \rightarrow (1) \)

(iv) \( \max \{-\mu(x), -\mu(y)\} = -\min \{\mu(x), \mu(y)\} \)

\[
\inf (-\mu(v)) = -\sup (\mu(v))
\]

Therefore, by definition of \((BFBZMV)\) and fuzzy BZMV algebra the proof is clear.

Hence proved.

**Lemma 3.1**: Let \( B_1 \) and \( B_2 \) are \((BFBZMV)\) of \( A \). Then \( B_1 \cap B_2 \) is a \((BFBZMV)\) of \( A \).

**Proof**: If \( x, y \in B_1 \cap B_2 \) then \( x, y \in B_1 \) and \( x, y \in B_2 \). Since \( B_1 \) and \( B_2 \) are \((BFBZMV)\), hence:

(i) \( \min \{ \mu^+_{B_1 \cap B_2}(x) \} = \min \{ \mu^+_{B_1}(x), \mu^+_{B_2}(x) \} \)

(ii) \( \min \{ \mu^-_{B_1 \cap B_2}(x) \} = \min \{ \mu^-_{B_1}(x), \mu^-_{B_2}(x) \} \)

For every \( y \in x \ominus y \)

(iii) \( \inf \{ \mu^+_{B_1 \cup B_2}(y) \} = \min \{ \inf \mu^+_{B_1}(y), \inf \mu^+_{B_2}(y) \} \)

\[ \geq \min \{ \mu^+_{B_1}(y), \mu^+_{B_2}(y) \} \]

\[
\inf \{ \mu^-_{B_1 \cup B_2}(y) \} = \max \{ \mu^-(x), \mu^-(y) \}
\]

\[ = \mu^-(0) \]

Hence \( \mu^-(a) \leq \mu^-(0) \) \( \rightarrow (2) \)

Now by, (1) and (2) and using Lemma we can conclude

\( \mu^+(a) = \mu^+(0) \) and \( \mu^-(a) = \mu^-(0) \).

Hence \( a \in S \) and this follows that \( x \ominus y \in A \).

Hence proved.

**Proposition 3.2**: Let \( S \) be a subset of \( A \) and \( B = (\mu^+; \mu^-) \) be a Bipolar valued fuzzy set determined as:

\[
\mu^+(x) = \begin{cases} k & \text{if } x \in S \\ 1 & \text{if } x \notin S \end{cases}
\]

\[
\mu^-(x) = \begin{cases} m & \text{if } x \in S \\ n & \text{if } x \notin S \end{cases}
\]

Where \( k, l \in [0, 1] \) and \( m, n \in [-1, 0] \) with \( k \leq 1, m \leq n \). Then \( B \) is a BFBZMV of \( A \) iff \( S \) is fuzzy BZMV algebra of \( A \).
Proof:
Let S be a fuzzy BZMV algebra of A. If x,y ∈ A are arbitrary hence:
If x ∈ S, then ¬ x ∈ S.
Hence μ^+(x) = μ^+(¬x), μ^−(x) = μ^−(¬x)
If x ∉ S, then ¬ x ∉ S.
Hence μ^+(x) = μ^+(¬x), μ^−(x) = μ^−(¬x)
we consider the following cases:

Case 1:
Let x, y ∈ S. Then x ⊕ y ⊆ S.
inf{μ^+(v)} = k
= min{k, k}
= min{μ^+(x), μ^+(y)}

and
sup{μ^+(v)} = m
= max{m, m}
= max{μ^−(x), μ^−(y)}

Case 2:
Let x, y ∉ S. Then,
μ^+(x) = 1 = μ^+(y)
μ^−(x) = n = μ^−(y)
so,
min{μ^+(x), μ^+(y)} = min{1, 1}
= 1
≤ inf {μ^+(v)}

Case 3:
Let x ∈ S and y ∉ S. Then:
μ^+(x) = k
μ^+(y) = 1
μ^−(x) = m
μ^−(y) = n

min{μ^+(x), μ^+(y)} = min{k, 1}
= 1
≤ inf {μ^+(v)}
max{μ^−(x), μ^−(y)} = max{m, n}
= n
= sup {μ^−(v)}

Hence B is a (BFBZMV) of A.

Conversely, Let x, y ∈ S.
Since B is (BFBZMV): μ^+(x) = k = μ^+(¬x)
Hence (¬x) ∈ S.
Now let a ∈ x ⊕ y
By the hypothesis we have:
inf{μ^+(v)} ≥ min{μ^+(x), μ^+(y)}
= min{k, k}
= k
and
sup{μ^−(v)} ≤ max{μ^−(x), μ^−(y)}
= max{m, m}
= m
Thus μ^+(a) = k and μ^−(a) = m. This follows a ∈ S, hence x ⊕ y ⊆ S and this proves that S is a fuzzy BZMV of A. Hence Proved.

Proposition 3.3:
If Bipolar valued fuzzy set B = (μ^+; μ^-) is a (BFBZMV)of A, then for all (s,t) ∈ [-1,0] × [0,1] the non empty strong positive t-cut of B and the non empty strong negative s-cut of B are fuzzy BZMV algebras of A.

Proof:
Let B = (μ^+; μ^-) be a (BFBZMV) of A and assume that B^+_t and B^-_r are non empty for all (s,t) ∈ [-1,0] × [0,1].
Let l, m ∈ B^+_t and p, q ∈ B^-_r. Then:
μ^+(l), μ^+(m) > 1
μ^- (p), μ^- (q) < s
Now,
Inf{ \mu^+(v)} \geq \min{\mu^+(l), \mu^+(m)} > t
\]
Where v \in x \oplus y, Which implies that l \oplus m \subseteq \beta I^+.

Now, let x \in \beta I^+.

Since B is a (BFMZMV) we have:
\mu^+ (\sim x) = \mu (x) > t
Hence \sim x and \sim x \in \beta I^+. So \beta I^+ is a BZMV algebra of A.

In other hand we have:
Sup \{ \mu^-(v) \} \leq \max\{ \mu^-(p), \mu^-(q)\} < s
Hence p \oplus q \in \beta I^-.
If x \in \beta I^- Since B is a (BFMZMV), then
\mu^- (\sim x) = \mu^- (x) < s
Hence \sim x \in \beta I^- and this proved that \beta I^- is a fuzzy BZMV algebra of A.
Hence Proved.

Corollary 3.1: If Bipolar valued fuzzy set B = (\mu^+: \mu^-) is a (BFMZMV) of A, then for all (s,t) \in [-1,0] \times [0,1] the non empty strong positive (s,t) - cut of B is a fuzzy BZMV algebra of A.

Proof:
We have, A_B^{(s,t)} = \beta I^+ \cap \beta I^-.
proof follows by proposition 3.3.
If Bipolar - valued fuzzy set B = (\mu^+: \mu^-) is a (BFMZMV) of A, then for all (s,t) \in [-1,0] \times [0,1] the non empty strong positive t - cut of B and the non-empty strong negative s-cut of B are fuzzy BZMV algebras of A.
Hence Proved.

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