Bipolar-valued fuzzy BZMV algebra

[1]EMIMANIMCY.M, [2]Dr. A. FRANCINA SHALINI

[1]PG SCHOLAR, [2]ASSISTANT PROFESSOR
DEPARTMENT OF MATHEMATICS,
NIRMALA COLLEGE FOR WOMEN,
COIMBATORE,TAMILNADU

Abstract: Brower-Zadeh MV – algebra is the result of a natural pasting between Brower – Zadeh algebras and MV – algebras. In this study the Bipolar – valued fuzzy values are introduced to Brower-Zadeh MV – algebras and the Strong s, t cuts are also defined.

Index Terms - BZMV- algebra, BZMV ^{dM}- algebra, Bipolar – valued fuzzy, strong cut

Introduction:

MV algebra have been introduced by C. CHANG [2]in order to provide an adequate semantic characterisation for Lukasiewicz many valued logics. (i.e)complete with respect to the evaluations of propositional variables in the real unit interval[0,1] Recall in fact that the prototypical example of an MV-algebra is the standard one [0,1]MV = < [0, 1], ⊕, ¬, 0 > where for all x,y ∈ [0,1],

$x \oplus y = \min\{1, x + y\}$, and $\neg x = 1 - x$.

In 1965, Zadeh introduced the notion of a fuzzy subset of a set. Since then it has become a vigorous area of research in different domains. There have been a number of generalizations of this fundamental concepts such as intuitionistic fuzzy sets, interval- valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0,1] to [-1,1]. In a bipolar valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree (0,1] indicates that elements somewhat satisfy the property, and the membership degree [-1,0) indicates that elements somewhat satisfy the implicit counter-property.

Bipolar-valued fuzzy sets and intuitionistic fuzzy sets look similar to each other. However, they are different from each other. Thus the bipolar-valued fuzzy set concepts are applied to the BZMV ALGEBRAS and some of the properties are verified.

Preliminaries:

[2] Definition1.1 :

A Brower Zadeh MV algebra (shortly BZMV algebra) is a structure A=< A,⊕, ¬, ∼, 0 > , where A is a non empty set of elements,0 is a constant element of A,¬and ∼ are unary operations on A, ⊕ is a binary operation on A. The following axioms hold:

- $(x\oplus y) \oplus z = (y\oplus z)\oplus x$ • $x\oplus 0 = x$
- $\neg(\neg x) = x$
- $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y)\oplus x$
- $x\oplus \sim\sim x = \neg 0$
- $x \oplus \sim\sim x = \sim\sim x$
- $\neg \sim [\neg(\neg x \oplus y) \oplus y] = \neg(\sim x\oplus \sim\sim y)\oplus \sim\sim y$

[17]Definition 1.2:

A de - Morgan Brower Zadeh MV algebra (shortly BZMV dM algebra) is a structure A = < A,⊕, ¬, ∼, 0 > that satisfies the axioms

- $(x \oplus y) \oplus z = (y\oplus z)\oplus x$
- $x \oplus 0 = x$
- $\neg(\neg x) = x$
- $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y)\oplus x$
- $\neg x\oplus \sim\sim x = \neg 0$
- $x \oplus \sim\sim x = \sim\sim x$
- $\neg \sim [\neg(\neg x \oplus y) \oplus y] = \neg(\sim x\oplus \sim\sim y)\oplus \sim\sim y$

and also the following condition:

$\sim \neg[\neg(x \oplus \neg y) \oplus \neg y] = \neg(\sim\sim x \oplus \neg \sim\sim y) \oplus \neg \sim\sim y$

Bipolar – valued fuzzy BZMV algebra:

Definition 2.1:[29]

Let G be a non empty set. A Bipolar-valued fuzzy set in G is an object having the form

$B = \{(x, \mu^+(x), \mu^-(x); x \in G\}$ where , $\mu^+ : G \to [0, 1]$ and $\mu^- : G \to [-1; 0]$ are mapping.

Note:

In this paper we use the symbol $B = (\mu^+; \mu^-)$ for the Bipolar-valued fuzzy set

$B = \{(x, \mu^+(x), \mu^-(x)); x \in G\}$

Definition 2.2:
A Bipolar-valued fuzzy set B = $(\mu^+; \mu^-)$ is called a "Bipolar-valued fuzzy BZMV -algebra" (BFBZMV) of M, if for every x, y in M it satisfies:
i) $\mu^+(x) \leq \mu^+(\neg x)$
ii) $\mu^-(x) \geq \mu^-(\neg x)$
iii) inf $\mu^+(x) \geq \min\{\mu^+(x), \mu^+(y)\}$
iv) sup $\mu^-(x) \leq \max\{\mu^-(x), \mu^-(y)\}$

Definition 2.3:
Let M be a nonempty set endowed with a operation ⊕, an unary operation ~ and ¬ and a constant 0 satisfying the following axioms,
for all x, y, z ∈ M:
- (x⊕y) ⊕ z = (y⊕z)⊕ x
- x⊕0 = x
- ¬(¬x) = x
- ¬(¬x ⊕ y) ⊕ y = ¬(x ⊕ ¬y)⊕x
- x⊕ ~~ x = ¬x
- x ⊕ ~~ x =~~x
- ¬ ~ [¬(¬x ⊕ y) ⊕ y] = ¬(~ x⊕ ~~ y)⊕ ~~y

For every subsets A and B of M we define the operators as follows:

$$a \oplus b = \begin{cases} a + b, \text{if } a+b < 1 \text{ and also if } a, b \in \mu^+ \\ 1 \ otherwise \\ a + b, \text{if } a+b > -1 \text{ and also if } a, b \in \mu^- \\ -1 \ otherwise \end{cases}$$

$$\neg a = \begin{cases} 1 - a, if\ a \in \mu^+ \\ -1 - a, if\ a \in \mu^- \end{cases}$$

$$\sim a = \begin{cases} 1, if\ a = 0\ and\ also\ a \in \mu^+ \\ -1, if\ a = 0\ and\ also\ a \in \mu^- \\ 0, \quad otherwise \end{cases}$$

for every a ∈ A and b ∈ B.

EXAMPLE 2.1:
Let M = < {0,a,1},¬, ~,0 > and define ⊕, ¬, ~ by the following tables:

$$a \oplus b = \begin{cases} a + b, \text{if } a+b < 1 \text{ and also if } a, b \in \mu^+ \\ 1 \ otherwise \\ a + b, \text{if } a+b > -1 \text{ and also if } a, b \in \mu^- \\ -1 \ otherwise \end{cases}$$

for every a ∈ A and b ∈ B.
The operator ¬ is defined as follows:
If a ∈ μ^+

X	0	a	1
¬x	1	1-a	0

If a ∈ μ^-

X	0	a	1
¬x	1	-1-a	0

The operator ~ is defined as follows:
If a ∈ μ^+

X	0	a	1
~ x	1	0	0

If a ∈ μ^-

X	0	A	1
~ x	-1	0	0

Now define μ^+ and μ^- as follows:
If a ∈ μ^+

X	0	a	1
μ^+(x)	0.7	0.2	0.7

If a ∈ μ^-

X	0	a	1
μ^-(x)	-1	-0.3	-1

Then M = < {0,a,1},¬, ~,0 > is a Bipolar – valued fuzzy BZMV algebra (BFBZMV) of M.

Proposition 2.1:
Let μ^+; μ^- be a BZMV algebra of M.
Then B = (μ^+; - μ^-) is a Bipolar-valued fuzzy BZMV algebra of M. Conversely, if B = (μ^+; -μ^-) is a Bipolar-valued fuzzy BZMV algebra of M, then (μ^+; μ^-) are BZMV algebra of M.

Proof:
Let x,y∈ M,
we know that ,
min{-$\mu(x)$, -$\mu(y)$}= −max{ $\mu(x)$, $\mu(y)$}
= min{$\mu^+_{B_1\cap B_2}(x)$, {$\mu^-_{B_1\cap B_2}(y)$}
and also,
$sup_{v\in x\oplus y}$ {$\mu^-_{B_1\cap B_2}(x)$}=max{$sup_{v\in x\oplus y}\mu^-_{B_1}(v), sup_{v\in x\oplus y}\mu^-_{B_2}(v)$}
≤ max{max{$\mu^-_{B_1}(x)$, $\mu^-_{B_1}(y)$ }, max{$\mu^-_{B_2}(x)$, $\mu^-_{B_2}(y)$}
= max {$\mu^-_{B_1\cap B_2}(x)$, $\mu^-_{B_1\cap B_2}(y)$}
Hence Proved.

Definition 2.4 :
Let A =< A, ⊕, ¬, ~, 0 > be BZMV algebra and S⊆ A be a non empty set containing ”0”. If S is a sub-structure BZMV algebra of A with respect to ”⊕” and ”¬” then we say that S is a MV algebra of A.
Lemma 2.1:
Let A =< A, ⊕, ¬, ~, 0 > be BZMV algebra and S⊆ A be a non empty set containing ”0”.Then S is a MV algebra of A iff for all x, y ∈S:
- x ⊕ y ∈ S
- ¬ x ∈ S

Definition 2.5 :
Let A =< A, ⊕, ¬, ~, 0 > be BZMV algebra and S⊆ A be a non empty set containing ”0”. If S is a sub-structure BZMV algebra of A with respect to ”⊕” and ”~” then we say that S is a MV algebra of A .
Lemma 2.2:
Let A =< A, ⊕, ¬, ~, 0 > be BZMV algebra and S⊆ A be a non empty set containing ”0”.Then S is a MV algebra of A iff for all x, y ∈S:
- x ⊕y ∈ S
- ~ x ∈ S

Definition 2.6:
A Bipolar-valued fuzzy set B = (μ^+; μ^-) is called a ”Bipolar-valued fuzzy MV - algebra” (BFMV) of A, if for every x, y ∈ A it satisfies:
i) $\mu^+(x)$ ≤ μ^+ (¬x)
ii) μ^- (x) ≥ μ^- (¬x)
iii) inf μ^+ (x) ≥ min{μ^+ (x), μ^+ (y)}
iv) sup μ^- (x) ≤ max{μ^- (x), μ^- (y)}

Remark 2.1 :
By the definition ¬(¬x) = x we have:
μ^+ (¬(¬ x)) ≥ μ^+ (¬x) ⇒ μ^+ (x)≥ μ^+ (¬x)
μ^- (¬x)≤ μ^- (¬(¬x)) ⇒ μ^- (¬x)≥ μ^- (x)
Hence conditions (i),(ii) in definition can be written as:
(i) μ^+ (x)= μ^+ (¬x)
ii) μ^- (x)= μ^- (¬x)

Strong positive t-cut and Strong negative s - cut
Definition 3.1:
Let B = (μ^+; μ^-) be a Bipolar-valued fuzzy set of A and
(s,t) ∈ [-1,0] ×[0,1].
Then:
- The set B_t^+= { x ∈ A; μ^+ (x) ≥t} is called positive t-cut of B.
- The set B_s^- = {x ∈ A; μ^- (x) ≤s} is called negative s-cut of B.

- The set ${}^{s}B_t^{+}$ = { x ∈ A; μ^{+} (x) >t} is called strong positive s-cut of B
- The set ${}^{s}B_s^{-}$ = {x ∈ A; μ^{-} (x) < s} is called strong negative s-cut of B.
- The set $A_B^{(t,s)}$= { x ∈ A; μ^{+} (x) ≥t , μ^{-} (x) ≤s } is called (t,s) cut of B.
- The set ${}^{s}A_B^{(t,s)}$ = { x ∈ A; μ^{+} (x) >t, μ^{-} (x) < s} is called strong (t,s) cut of B.

Note that $A_B^{(t,s)} = B_t^{+} \cap B_s^{-}$

Proposition 3.1:

Let B = (μ^{+}; μ^{-}) be a (BFBZMV) of A. Then the set,
A_B= { x ∈ A; μ^{+} (x) = μ^{+} (0), μ^{-} (x) = μ^{-} (0)} is a fuzzy BZMV algebra of A.

Proof:

Let x, y ∈ A_B
Then μ^{+} (x) = μ^{+} (0) = μ^{+} (y) and
μ^{-} (x) = μ^{-} (0) = μ^{-} (y).
Thus,
(i) μ^{+} (¬ x) = μ^{+} (x) = μ^{+} (0)
(ii) μ^{-} (¬ x) = μ^{-} (x) = μ^{-} (0)
Hence ¬ x ∈ A_B

Now let a ∈ x⊕ y ,
Then (iii) $inf_{v\in x\oplus y}\mu^{+}$ (v) ≥ min{μ^{+} (x), μ^{+} (y)}
= min{μ^{+} (0), μ^{+} (0)}
= μ^{+} (0)
Hence μ^{+} (a) ≥ μ^{+} (0) → (1)

(iv) max {-μ(x) ,-μ(y)} = - min {μ(x) ,μ(y)}
Inf (- μ(v)) = - sup (μ(v))
sup (- μ(v)) = - inf (μ(v))
Hence by definition of (BFBZMV) and fuzzy BZMV algebra the proof is clear.
Hence Proved

Lemma 3.1 :

Let B_1 and B_2 are (BFBZMV) of A . Then $B_1 \cap B_2$ is a (BFBZMV) of A.

Proof:

If x,y ∈ $B_1 \cap B_2$ then x, y ∈ B_1 and x, y ∈ B_2 .Since B_1 and B_2 are (BFBZMV), hence:
(i) min {$\mu^{+}_{B_1\cap B_2}(x)$} = min{$\mu^{+}_{B_1}(x)$, $\mu^{+}_{B_2}(x)$}
= min{$\mu^{+}_{B_1}$(¬x), $\mu^{+}_{B_2}$ (¬x)}
= min{$\mu^{+}_{B_1\cap B_2}$ (¬x)}
(ii) min{$\mu^{-}_{B_1\cap B_2}(x)$} = min {$\mu^{-}_{B_1}(x)$, $\mu^{-}_{B_2}(x)$}
= min {$\mu^{-}_{B_1}$(¬x), $\mu^{-}_{B_2}$ (¬x)}
= min {$\mu^{-}_{B_1\cap B_2}$ (¬x)}
for every v ∈ x ⊕ y
(iii) inf {$\mu^{+}_{B_1\cap B_2}$ (v)} = min{ inf $\mu^{+}_{B_1}$ (v), inf $\mu^{+}_{B_2}$ (v)}
≥ min{min{$\mu^{+}_{B_1}$ (x), $\mu^{+}_{B_1}$ (y)}, min{$\mu^{+}_{B_2}$ (x), $\mu^{+}_{B_2}$ (y)}}
(iv) sup{$\mu^{-}_{B_1\cap B_2}$ (v) ≤ max{μ^{-} (x), μ^{-} (y)}
= max{μ^{-} (0), μ^{-} (0)}
= μ^{-} (0)
Hence μ^{-} (a) ≤ μ^{-} (0) →(2)
Now by, (1) and (2) and using lemma we can conclude
μ^{+} (a) = μ^{+} (0) and μ^{-} (a) = μ^{-} (0).
Hence a ∈ S
and this follows that x ⊕ y ⊆ A.
Hence Proved.

Proposition 3.2 :

Let S be a subset of A and B = (μ^{+}; μ^{-}) be a Bipolar -valued fuzzy set determined as:

$$\mu^{+}(x) = \begin{cases} k \text{ if } x \in S \\ l \text{ if } x \notin S \end{cases}$$

$$\mu^{-}(x) = \begin{cases} m \text{ if } x \in S \\ n \text{ if } x \notin S \end{cases}$$

Where k,l ∈ [0,1] amd m,n ∈ [-1,0] with k≥ l, m≤ n. Then B is a BFBZMV of A iff S is fuzzy BZMV algebra of A.

Proof:

Let S be a fuzzy BZMV algebra of A.

If x,y ∈ A are arbitrary hence:

If x ∈ S, then ¬ x ∈ S.

Hence μ^{+}(x) = μ^{+} (¬ x), μ^{-} (x) = μ^{-} (¬x)

If x ∉ S, then ¬ x ∉ S.

Hence μ^{+}(x) = μ^{+} (¬x), μ^{-} (x) = μ^{-} (¬x)

we consider the following cases:

Case 1:

Let x,y ∈ S. Then x ⊕ y ⊆ S.

inf{μ^{+}(v)} = k

= min{k, k}

= min{μ^{+} (x), μ^{+} (y)}

and sup{μ^{+}(v)} = m

= max{m, m}

= max{μ^{-} (x), μ^{-} (y)}

Case 2:

Let x,y ∉ S. Then,

μ^{+} (x) = l = μ^{+} (y)

μ^{-} (x) = n = μ^{-} (y)

so, min{μ^{+} (x), μ^{+} (y)} = min{l, l}

= l

≤ inf { μ^{+} (v)}

Case 3:

Let x∈ S and y ∉ S.Then:

μ^{+} (x) = k μ^{+} (y) = l

μ^{-} (x)= m μ^{-} (y) = n

min{ μ^{+} (x), μ^{+} (y)} = min{k, l}

= l

≤ inf { μ^{+} (v)}

max{ μ^{-} (x), μ^{-} (y)} = max{m, n}

= n

= sup { μ^{-} (v)}

Hence B is a (BFBZMV) of A.

Conversely,

Let x,y ∈ S.

Since B is (BFBZMV): μ^{+} (x) = k = μ^{+} (¬x)

Hence (¬x)∈ S.

Now let a ∈ x ⊕ y

By the hypothesis we have:

inf{ μ^{+}(v)} ≥ min{ μ^{+} (x), μ^{+} (y)}

= min {k, k}

= k

and sup{ μ^{-}(v)} ≤ max{μ −(x), μ−(y)}

= max {m, m}

= m

Thus μ^{+} (a) = k and μ^{-} (a)= m. This follows a ∈ S, hence x ⊕ y ⊆ S and this proves that S is a fuzzy BZMV of A.

Hence Proved.

Proposition 3.3 :

If Bipolar - valued fuzzy set B = (μ^{+}; μ^{-}) is a (BFBZMV)of A, then for all (s,t) ∈ [-1,0] × [0,1] the non empty strong positive t- cut of B and the non empty strong negative s-cut of B are fuzzy BZMV algebras of A.

Proof:

Let B = (μ^{+}; μ^{-}) be a (BFBZMV) of A and assume that ${}^{s}B_t^{+}$ and ${}^{s}B_t^{-}$ are non empty for all (s,t) ∈ [-1,0] × [0,1].

Let l , m ∈ ${}^{s}B_t^{+}$ and p , q ∈ ${}^{s}B_t^{-}$. Then:

μ^{+} (l), μ^{+} (m) > l

μ^{-} (p), μ^{-}(q) < s

Now,

Inf{ μ^+ (v)} ≥ min{ μ^+ (l), μ^+ (m)}
> t
Where v ∈ x ⊕y , Which implies that l ⊕ m ⊆ ${}^sB_t^+$.
Now , let x ∈ ${}^sB_t^+$.
Since B is a (BFBZMV) we have:
μ^+ (¬ x) =μ (x) > t
Hence ¬x and ~x ∈ ${}^sB_t^+$. So ${}^sB_t^+$ is a BZMV algebra of A.
In other hand we have:
Sup { μ^- (v)} ≤ max{ μ^- (p), μ^- (q)}
< s
Hence p ⊕ q ∈ ${}^sB_t^-$.
If x ∈ ${}^sB_t^-$. Since B is a (BFBZMV) , then
μ^- (¬ x) = μ^- (x) < s
Hence ¬x ∈ ${}^sB_t^-$ and ~ x ∈ ${}^sB_t^-$ and this proved that ${}^sB_t^-$ is a fuzzy BZMV algebra of A.
Hence Proved.

Corollary 3.1:
If Bipolar valued fuzzy set B = (μ^+; μ^-) is a (BFBZMV) of A,then for all
(s,t) ∈ [-1,0] × [0,1] the non empty strong positive (s,t) - cut of B is a fuzzy BZMV algebra of A.
Proof:
We have, $A_B^{(t,s)}$ = ${}^sB_t^+$ ∩ ${}^sB_t^-$.
proof follows by proposition 3.3,

If Bipolar - valued fuzzy set B = (μ^+; μ^-) is a (BFBZMV)of A, then for all (s,t) ∈ [-1,0] × [0,1] the non empty strong positive t- cut of B and the non-empty strong negative s-cut of B are fuzzy BZMV algebras of A.

Hence Proved.

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