GENERALISED INNER DERIVATIONS

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Abstract. Let A be any ring and f be a generalised inner derivation of A. For any $x,y \in A$, we have $f(xy) = f(x)y + xh_a(y)$ for fixed element $a \in A$.
In this paper, it is shown that (i) If $r\,f(x) = 0$ for every $r,x \in A$, A = Prime ring. Then either $r = 0$ or $h_a = 0$ (ii) $f(xyz) = f(xy)z + xf(yz) - xf(y)z\ \forall\ x,y,z \in A$ (iii) $f(aba) = f(a)ba\ \forall\ b \in A$.
We proved Posner [4] Lemma 1 P.1093, Bresar [1] Obs.1, Remark 3, P.90, Havala [2] Def. P1147 as Corollaries along with other results.

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Introduction

We define the generalised inner derivations of a ring. "Let A be any ring. An additive mapping $f : A \to A$ is said to be generalised inner derivation if $f(xy) = f(x)y + xh_a(y)$
where
$h_a : A \to A\ \forall\ x,y \in A$ & fixed element $a \in A$ $y \to h_a(y)$.
Let $G_ID(A)$ be the set of all generalised inner derivations of A into itself. In Thm. 2.1 we have proved that if $rf(x) = 0\ \forall\ x \in A$ then either $r = 0$ or $h_a = 0$ where $f(xy) = f(x)y + xh_a(y)\ \forall\ x,y \in A$, fixed element $a \in A$ and f be the generalised inner derivation on a Prime ring A. In Cor. 2.3, Replacing h_a by d, we get Lemma 1, P.1093 of Posner [4]. In Thm. 2.4 we have proved that if $f \in G_ID(A)$, A be any Prime ring. Then $f(xyz) = f(xy)z + xf(yz) - xf(y)z\ \forall\ x,y,z \in R$. In Cor.2.5, replacing f by d we get Obs.1, Remark 3 on P.90 of Bresar [1]. In Cor. 2.6 if A has unity then generalised inner derivations becomes the inner derivation and vice versa. In Thm. 2.7, we have proved that if A is Prime ring and f is generalised inner derivation of A then $f(aba) = f(a)ba\ \forall\ b \in A$, a = fixed element of A. In Lemma 2.8, let A be any Prime ring and $f \in G_ID(A)$ s.t. $xf(x) - f(x)x = 0\ \forall\ x \in A$ Then $x(f(x)a + af(x)) = f(x)(ax + xa)$, for fixed element $a \in A$. In 3.1, replacing h_a by d, we get Havala [2] definition of Generalised derivation. In Thm. 3.3, we have proved that if f is generalised derivation of A and if a 6= $0 \in A$. Then $d(b)a = ad(b)\ \forall\ b \in A$.
In the last, we have proved that if f is generalised derivations of A, A is without zero Divisors and $ab = 0$. Then $f(ba) = f(a)b + f(b)a\ \forall\ a,b \in A$.

1. Generalised Inner Derivations

In this section, we study the Generalised Inner Derivation in a Ring. Let A be any Ring.
Definition 1.1 (Generalised Inner Derivation): An additive mapping $f : A \to A$ is said to be Generalized Inner derivation if $f(xy) = f(x)y + x[a,y]$, for fixed element $a \in A\ \forall\ x,y \in A$
We are taking the definition as

$$f(xy) = f(x)y + xh_a(y)$$

where
$h_a :$ $A \to A$
$y \to h_a(y) = [a,y]$ is inner derivation
Let $G_ID(A)$ = set of all generalised inner derivation of A into itself.

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In this section, we take A be any Prime ring.
Definition 2.1 (Prime Ring): Let A be any ring. Then A is said to be Prime Ring iff
$xay = 0\ \forall\ a \in A$
$\Rightarrow x = 0$ or $y = 0$
Lemma 2.2 Let f be a generalised inner derivation of a Prime ring and $r \in A$.
If $rf(x) = 0\ \forall\ x \in A$ Then either $r = 0$ or $h_a = 0$ where $f(xy) = f(x)y + xh_a(y)$, $\forall\ x,y \in A$ & fixed element $a \in A$
Proof. Now $rf(x) = 0\ \forall\ x \in A$. Replacing x by xy, we get $rf(xy) = 0$
$\Rightarrow r(f(x)y + xh_a(y)) = 0$
$\Rightarrow rf(x)y + rxh_a(y) = 0$
$\Rightarrow\ rxh_a(y) = 0$
$\Rightarrow$either $r = 0$ or $h_a(y) = 0$($\because A$ is Prime Ring)
$\Rightarrow$either $r = 0$ or $h_a = 0$
Hence proved.

Corollary 2.3 Replacing h_a by d, we have Lemma 1, P.1093 of Posner [4].
Theorem 2.4 If $f \in G_ID(A)$ where A is Prime Ring. Then $f(xyz) = f(xy)z + xf(yz) - xf(y)z\ \forall\ x,y,z \in A$

Proof. Now $f(xyz) - f(xy)z - xf(yz) + xf(y)z$

$= \{f(xy)z + xyh_a(z)\} - f(xy)z - x\{f(y)z + yh_a(z)\} + xf(y)z$

$= f(xy)z + xyh_a(z) - f(xy)z - xf(y)z - xyh_a(z) + xf(y)z$

$= 0.$

Hence $f(xyz) = f(xy)z + xf(yz) - xf(y)z \ \forall\ x,y,z \in A$. Hence proved.

Corollary 2.5 Replacing f by d, we get Obs.1, Remark 3, P.90 of Bresar [1].

Corollary 2.6 If A has unity. Then Generalised inner derivations become inner derivations and vice-versa.

Proof. $f(xy) = f(x)y + xh_a(y)$ for fixed element $a \in A\ \forall\ x,y \in A$.

Putting $x = 1$, we have

$$f(y) = f(1)y + 1h_a(y)$$

$$\Rightarrow f(y) = f(1)y + h_a(y)$$

$\Rightarrow$ f becomes inner derivations and vice versa

Hence proved.

Theorem 2.7 If A is a Prime Ring and f is generalised inner derivations of A.

Then $f(aba) = f(a)ba\ \forall\ b \in A$ where a be the fixed element of A.

Proof. Since f is generalised inner derivation of A

$\Rightarrow f(xy) = f(x)y + xh_a(y)$ $\forall\ x,y \in A$&fixed element $a \in A$

Now

$$f(a^2) = f(a)a + ah_a(a) = 0$$

$$f(a^2) = f(a)a$$

Replacing a by $a + b$, we get $f(a + b)^2 = f(a + b)(a + b)$

$\Rightarrow f(a^2) + f(ab + ba) + f(b^2) = (f(a) + f(b))(a + b)$

$\Rightarrow f(ab + ba) = f(b)a + f(a)b$

Let

$K = f(a(ab + ba) + (ab + ba)a)$

$= f(a)(ab + ba) + f(ab + ba)a$

$\Rightarrow K = f(a)ab + f(a)ba + (f(b)a + f(a)b)a$

$= f(a)ab + f(a)ba + f(b)a^2 + f(a)ba$ (1)

On the other hand

$K = f(a^2b + 2aba + ba^2)$

$K = f(a^2b + ba^2) + 2f(aba)$

$= (f(a^2)b + f(b)a^2) + 2f(aba)$

$= f(a)ab + f(b)a^2 + 2f(aba)$ From (1) and (2) we have (2)

$2f(aba) = 2f(a)ba$

$\Rightarrow f(aba) = f(a)ba$

Hence proved.

Lemma 2.8 Let A be any Prime Ring and $f \in G_lD(A)$ such that $xf(x) - f(x)x = 0\ \forall\ x \in A$. Then $x(f(x)a + af(x)) = f(x)(ax + xa)$, for fixed element $a \in A$.

Proof. $xf(x) - f(x)x = 0$ Replacing x by $x + y$, we get

$$\left.\begin{aligned} xf(y) - f(x)y &= f(y)x - yf(x) \\ f(x)y + xh_a(y) &= f(xy) \end{aligned}\right]$$ Adding both, we get

$$xf(y) + xh_a(y) = f(y)x - yf(x) + f(xy)$$

Replacing y by xa

$$xf(xa) + xh_a(xa) = f(xa)x - xaf(x) + f(x^2a)$$

$$\Rightarrow xf(x)a + xh_a(x)a = f(x)ax - xaf(x) + f(x^2)a$$

$$\Rightarrow xf(x)a + xaf(x) = f(x)ax + f(x)xa$$

$\Rightarrow x(f(x)a + af(x)) \quad = \quad f(x)(ax + xa)$

Hence proved.

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In this section, we take A be any Prime Ring of ch 6= 2

3.1 By replacing h_a by d, we get Havala [2] definition

$$f(xy) = f(x)y + xd(y) \ \forall \ x,y \in A$$

Lemma 3.2 Let A be any ring. Let

$$T(a) = \{r \in A; r(ax - xa) = 0 \ \forall \ x \in A\} \text{ for any } a \in A$$

Then $T(a)$ is a Two Sided Ideal of A. By using 3.1 and 3.2, we will prove the following theorem.

Theorem3.3 If f is a generalised derivation of A and if a 6= $0 \in A$ Then $d(b)a = ad(b) \ \forall \ b \in A$.

Proof. Now

$$f(aba) = f(a)ba + ad(b)a + abd(a) \qquad (3)$$

Also

$$f(aba) = f(a)ba + ad(ba)$$

$$= f(a)ba + a(bd(a) + ad(b))$$

$$= f(a)ba + abd(a) + a^2d(b) \qquad (4)$$

From (3) and (4) we have $ad(b)a = a^2d(b)$

$$\Rightarrow \quad a(d(b)a - ad(b)) = 0$$

By Lemma 3.2, $a \in T(a)$ and $T(a)$ is two sided Ideal of A, a 6= 0

$\Rightarrow \quad d(b)a - ad(b) = 0$

$\Rightarrow \quad d(b)a + ad(b)$

Hence proved.

Lemma 3.4 If f is generalised derivation of A. Then

$$f(ab + ba) = f(a)b + f(b)a + d(ab) \ \forall \ a,b \in A$$

Lemma 3.5 If f is generalised derivation of A. Then

$$f(aba) = f(a)ba + ad(b)a + abd(a) \ \forall \ a,b \in A$$

Remark Above Lemmas 3.4 and 3.5 can also be proved by taking $f(c^2) = f(c)c+cd(c)$ & $c = a+b$ and $K = f(a(ab+ba)+(ab+ba)a)$. But this course of action would be quite lengthy.

Lemma 3.6 If f is generalised derivation of A. Then

$f(abc + cba) = f(a)bc + f(c)ba + ad(b)c + abd(c) + cd(b)a + cbd(a) \ \forall \ a,b,c \in A.$

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In this section, we take A is without zero divisors.

Theorem 4.1 If f is generalised derivation of A, A is without zero divisors and if $ab = 0$ then $f(ab) = f(a)b + f(b)a \ \forall \ a,b \in A$.

Proof. By Lemma 3.4, we have

$$f(ab + ba) = f(a)b + f(b)a + d(ab)$$

Putting $ab = 0$, we get

$$f(ba) = f(a)b + f(b)a + d(0)$$

Since A is without zero divisors

$\Rightarrow d(0) \quad = \quad 0$

$\Rightarrow f(ba) \quad = \quad f(a)b + f(b)a.$

Hence proved.

Conclusion

After, giving the basic introduction to the generalised inner Derivation to a ring we established the results that are used to prove the most fundamental results Posner [4] Lemma 1 P.1093, Bresar [1] Obs.1, Remark 3, P.90, Havala [2] Def. P1147 as corollaries.

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