Expansion of The Quadruple Hypergeometric Polynomial Set $M_n(x_1, x_2, x_3, x_4)$ in Terms of The Jacobi Polynomial

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Abstract: **In this paper, an effort has been done to give Expansion of The Quadruple Hypergeometric Polynomial set $M_n(x_1, x_2, x_3, x_4)$ in Terms of The Jacobi Polynomials. Many interesting new results may be obtained on specializing the respective parameters in which some of them are believed to be new.**

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I. Introduction:

M. Kumar and B.K. Singh [3] defined the quadruple hypergeometric polynomial set $M_n(x_1, x_2, x_3, x_4)$ by means of the generating relation.

$$(\xi-\lambda t)^{\sigma} F\begin{bmatrix} \lambda_1,(\alpha_{u_1}); \\ \mu_1 x_1^{-e_1} t^{e_1} \\ (\beta_{v_1}); \end{bmatrix} \times F\begin{bmatrix} (A_p);(C_r);(\alpha_{u_2});(\alpha_{u_3});(\alpha_{u_4}) \\ \mu x_1^{e} t, \mu_2 x_2^{-e_2} t^{e_2}, \mu_3 x_3^{e_3} t^{e_3}, \mu_4 x_4^{e_4} t^{4} \\ (B_q);(D_s);(\beta_{v_2});(\beta_{v_3});(\beta_{v_4}) \end{bmatrix}$$

$$= M^{\lambda;\lambda_1;\xi_1;\sigma;\mu;\mu_1;\mu_2;\mu_3;\mu_4;(A_p);(C_r);(\alpha_{u_1});(\alpha_{u_2});(\alpha_{u_3});(\alpha_{u_4})}_{n,e;e_1;e_2;e_3;e_4;(B_q);(D_s);(\beta_{v_1});(\beta_{v_2});(\beta_{v_3});(\beta_{v_4})}(x_1, x_2, x_3, x_4)t^n \qquad \ldots (1.1)$$

where □, □$_1$, □,□□,□□,□□$_1$,□□$_2$, □$_3$, □$_4$ are real and e, e_1, e_2, e_3, e_4 are non-negative integers.

The left hand side of (1.1) contains the product of generalized hypergeometric function and Lauricella function in the notation of Burchnall and Chaundy [2]. The polynomial set contains number of parameters. For simplicity we shall denote

$M^{\lambda;\lambda_1;\xi_1;\sigma;\mu;\mu_1;\mu_2;\mu_3;\mu_4;(A_p);(C_r);(\alpha_{u_1});(\alpha_{u_2});(\alpha_{u_3});(\alpha_{u_4})}_{n,e;e_1;e_2;e_3;e_4;(B_q);(D_s);(\beta_{v_1});(\beta_{v_2});(\beta_{v_3});(\beta_{v_4})}(x_1, x_2, x_3, x_4)$ by $M_n(x_1, x_2, x_3, x_4)$.

where n is the order of the polynomial set.

After little simplification (1.1) gives

$$M_n(x_1, x_2, x_3, x_4) = \sum_{h=0}^{n} \sum_{h_1=0}^{\left[\frac{n-h}{e_1}\right]} \sum_{h_2=0}^{\left[\frac{n-h-e_1h_1}{e_2}\right]} \sum_{h_3=0}^{\left[\frac{n-h-e_1h_1-e_2h_2}{e_3}\right]} \sum_{h_4=0}^{\left[\frac{n-h-e_1h_1-e_2h_2-e_3h_3}{e_4}\right]}$$

$$\times \frac{[(A_p)]_{n-h-e_1h_1-(e_2-1)h_2-(e_3-1)h_3-(e_4-1)h_4}}{[(B_q)]_{n-h-e_1h_1-(e_2-1)h_2-(e_3-1)h_3-(e_4-1)h_4}}$$

$$\times \frac{[(C_r)]_{n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4} [(\alpha_{u_1})]_{h_1} [(\alpha_{u_2})]_{h_2} [(\alpha_{u_3})]_{h_3} [(\alpha_{u_4})]_{h_4}}{[(D_s)]_{n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4} [(\beta_{v_1})]_{h_1} [(\beta_{v_2})]_{h_2} [(\beta_{v_3})]_{h_3} [(\beta_{v_4})]_{h_4}}$$

$$\times \frac{(\sigma)_h \lambda^h (\lambda_1)_{h_1} \; \mu_1^{h_1} \; \mu_2^{h_2} \; (\mu_3 x_3^{e_3})^{h_3}}{\xi^{\sigma+h} \; h! \; h_1! \; x_2^{e_1h_1+e_2h_2}! \; h_3!}$$

$$\times \frac{\left(\mu_4 x_4^{e_4}\right)^{h_4} \quad \left(\mu x_1^{e}\right)^{n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4}}{h_4!\ \left(n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4\right)!} \qquad \ldots (1.2)$$

II. Notations:

1. (i) $(n) = 1, 2, \ldots\ldots n-1, n.$

(ii) $(a_p) = a_1, a_2, a_3, \ldots\ldots a_p.$

(iii) $(a_p; i) = a_1, a_2, a_3, \ldots\ldots a_{i-1}, a_{i+1} \ldots\ldots a_p.$

2. (i) $[(a_p)] = a_1, a_2, a_3 \ldots\ldots a_p.$

(ii) $$\left[\left(a_p\right)\right]_n = \prod_{i=1}^{p}\left(a_i\right)_n = \left(a_1\right)_n\left(a_2\right)_n\left(a_3\right)_n\ldots\left(a_p\right)_n$$

(iii) $$\left[\left(a_p\right)+m\left(p\right)\right]_n = \prod_{i=1}^{p}\left(a_i+m_i\right).$$

3. (i) $$\Delta\left(a;b\right)=\frac{b}{a},\frac{b+1}{a},\frac{b+2}{a}+\ldots\ldots\frac{b+a-1}{a}.$$

(ii) $$\Delta\left(a\left(1\right);b\right)=\frac{b}{a},\frac{b+1}{a},\frac{b+2}{a},\ldots\ldots\frac{b+a-2}{a}$$

(iii) $$\Delta\left(m;\left(a_p\right)\right)=\left(\frac{a_1+r-1}{m}\right),\ \begin{matrix} r=1;\ldots\ldots m \\ i=1;\ldots\ldots p\end{matrix}$$

(iv) □$(a; b \pm c \pm d)$= □$(a; b+c+d)$, □$(a; b+c-d)$, □$(a; b-c+d)$, □$(a; b-c-d)$,

4. (i) $$\Delta_k\left[a;b\right]=\prod_{r=1}^{a}\left(\frac{b+r-1}{a}\right)_k=\left(\frac{b}{a}\right)_k\left(\frac{b+1}{a}\right)_k\ldots\left(\frac{b+a-1}{a}\right)_k.$$

(ii) $$\Delta_k\left[a\left(1\right);b\right]=\left(\frac{b}{a}\right)_k\left(\frac{b+1}{a}\right)_k\ldots\left(\frac{b+a-2}{a}\right)_k.$$

(iii) $$\Delta_k\left[m\left(a_p\right)\right]=\prod_{i=1}^{p}\prod_{r=1}^{a}\left(\frac{a_i+r-1}{m}\right)_k.$$

5. (i) $$\Gamma\left[\left(a_p\right)\right]=\prod_{i=1}^{p}\left(a_i\right).$$

(ii) $$\Gamma\left[a+\frac{(m)}{m}\right]=\prod_{r=1}^{m}\left(a+\frac{r}{m}\right).$$

(iii) $$\Gamma\left[\left(a,b\right)\right]=\prod_{r=1}^{a}\Gamma\left(\frac{b+r-1}{a}\right)$$

(iv) $$\Gamma\left[\Delta\left(m\right);\left(a_p\right)\right]=\prod_{i=1}^{p}\prod_{r=1}^{m}\Gamma\left(\frac{a_i+r-1}{m}\right)$$

6. (i) $$\Gamma_*\left(a\pm b\right)=\Gamma\left(a+b\right)\Gamma\left(a-b\right).$$

(ii) $$\Gamma_{**}\left(a+b\right)=\Gamma\left(a+b\right)\Gamma\left(a+b\right)$$

$$K=\frac{\left[\left(A_p\right)\right]_n\left[\left(C_r\right)\right]_n\left(\mu x_1^{e}\right)^n}{\left[\left(B_q\right)\right]_n\left[\left(D_k\right)\right]_n\xi^{\sigma}n!}$$

III. $M_n(x_1, x_2, x_3, x_4)$ in Terms of The Jacobi Polynomials :

We have from (1.2)

$$\sum_{n=0}^{n} M_n\left(x_1,x_2,x_3,x_4\right)=\sum_{n=0}^{\infty}\sum_{h,h_1,h_2,h_3,h_4=0}^{\infty}\frac{\left[\left(A_p\right)\right]_{n+h_2+h_3+h_4}}{\left[\left(B_q\right)\right]_{n+h_2+h_3+h_4}}$$

$$\times\frac{\left[\left(C_r\right)\right]_n\left[\left(\alpha_{u_1}\right)\right]_{h_1}\left[\left(\alpha_{u_2}\right)\right]_{h_2}\left[\left(\alpha_{u_3}\right)\right]_{h_3}\left[\left(\alpha_{u_4}\right)\right]_{h_4}(\sigma)_h\,\lambda^h\left(\lambda_1\right)_{h_1}\mu_1^{h_1}}{\left[\left(D_s\right)\right]_n\left[\left(\beta_{v_1}\right)\right]_{h_1}\left[\left(\beta_{v_2}\right)\right]_{h_2}\left[\left(\beta_{v_3}\right)\right]_{h_3}\left[\left(\beta_{v_4}\right)\right]_{h_4}\xi^{\sigma+h}\ h!\ h_1!}$$

$$\times\frac{\mu_2^{h_2}\left(\mu_3x_3^{e_3}\right)^{h_3}\left(\mu_4x_4^{e_4}\right)^{h_4}\mu^n\left(x_1^e\right)^n t^{n+h+e_1h_1+e_2h_2+e_3h_3+e_4h_4}}{h_2!\ x_2^{e_1h_1+e_2h_2}\ h_3!\ h_4!\ n!} \qquad \ldots (3.1)$$

Also, we obtain from [1]

$$\left(x_1^e\right)^n=n!(x+c)!=\sum_{i=0}^{\infty}\frac{(-1)^j\left(2i++c+d+1\right)P_i^{(c,d)}\left(1+2x_1^e\right)}{(n-i)!(c+d+i+1)_{n+1}}$$

Hence, (3.1) can be written as

$$\sum_{n=0}^{\infty} M_n\left(x_1,x_2,x_3,x_4\right)=\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{h,h_1,h_2,h_3,h_4=0}^{\infty}\frac{\left[\left(A_p\right)\right]_{n+h_2+h_3+h_4}}{\left[\left(B_q\right)\right]_{n+h_2+h_3+h_4}}$$

$$\times\frac{\left[\left(C_r\right)\right]_n\left[\left(\alpha_{u_1}\right)\right]_{h_1}\left[\left(\alpha_{u_2}\right)\right]_{h_2}\left[\left(\alpha_{u_3}\right)\right]_{h_3}\left[\left(\alpha_{u_4}\right)\right]_{h_4}(\sigma)_h\,\lambda^h\left(\lambda_1\right)_{h_1}\mu_1^{h_1}}{\left[\left(D_s\right)\right]_n\left[\left(\beta_{v_1}\right)\right]_{h_1}\left[\left(\beta_{v_2}\right)\right]_{h_2}\left[\left(\beta_{v_3}\right)\right]_{h_3}\left[\left(\beta_{v_4}\right)\right]_{h_4}\xi^{\sigma+h}\ h!\ h_1!}$$

$$\times\frac{\mu_2^{h_2}\left(\mu_3x_3^{e_3}\right)^{h_3}\left(\mu_4x_4^{e_4}\right)^{h_4}\mu^n(-1)^i\left(2i+c+d+1\right)}{h_2!\ x_2^{e_1h_1+e_2h_2}\ h_3!\ h_4!\ (c+d+i+1)_{n+1}}$$

$$\times\frac{(n+c)!P_i^{(c,d)}\left(1+2x_1^e\right)\left(x_1^e\right)^n t^{n+h+e_1h_1+e_2h_2+e_3h_3+e_4h_4}}{(n-i)!}$$

$$=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{h=0}^{n}\sum_{h_1=0}^{\left[\frac{n-h}{e_1}\right]}\sum_{h_2=0}^{\left[\frac{n-h-e_1h_1}{e_2}\right]}\sum_{h_3=0}^{\left[\frac{n-h-e_1h_1-e_2h_2}{e_3}\right]}\sum_{h_4=0}^{\left[\frac{n-h-e_1h_1-e_2h_2-e_3h_3}{e_4}\right]}$$

$$\times\frac{\left[\left(A_p\right)\right]_{n-h-e_1h_1-(e_2-1)h_2-(e_3-1)h_3-(e_4-1)h_4}}{\left[\left(B_q\right)\right]_{n-h-e_1h_1-(e_2-1)h_2-(e_3-1)h_3-(e_4-1)h_4}}$$

$$\times\frac{\left[\left(C_r\right)\right]_{n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4}\left[\left(\alpha_{u_1}\right)\right]_{h_1}\left[\left(\alpha_{u_2}\right)\right]_{h_2}\left[\left(\alpha_{u_3}\right)\right]_{h_3}\left[\left(\alpha_{u_4}\right)\right]_{h_4}}{\left[\left(D_s\right)\right]_{n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4}\left[\left(\beta_{v_1}\right)\right]_{h_1}\left[\left(\beta_{v_2}\right)\right]_{h_2}\left[\left(\beta_{v_3}\right)\right]_{h_3}\left[\left(\beta_{v_4}\right)\right]_{h_4}}$$

$$\times\frac{(\sigma)_h\,\lambda^h\left(\lambda_1\right)_{h_1}\mu_2^{h_2}\ldots\left(\mu_3x_3^{e_3}\right)^{h_3}\left(\mu_4x_4^{e_4}\right)^{h_4}}{\xi^{\sigma+h}\ h!\ h_1!\ h_2!\ x_2^{e_1h_1+e_2h_2}\ h_3!\ h_4!}$$

$$\times\frac{\mu^{n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4}(-1)^i\left(2i+c+d+1\right)P_i^{(c,d)}\left(1+2x_1^e\right)}{\left(n-i-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4\right)!}$$

$$\times\frac{n-c-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4\ t^n}{(c+d+i+1)_{n+1-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4}} \qquad \ldots (3.2)$$

Equating the co-efficient of t^n from both sides and after little simplification, we finally achieve for $e_2 > 1$, $e_3 > 1$ and $e_4 > 1$

$$M_n(x_1,x_2,x_3,x_4) = K\sum_{i=0}^{n} \frac{(n+c)!(-1)^i\,(2i+c+d+1)\,P_i^{(c,d)}\left(1+2x_1^e\right)}{(n-i)!(c+d+i+1)_{n+1}}$$

$$\times \sum_{h,h_1,h_2,h_3,h_4=0}^{\infty} \frac{\left[1-(B_q)-n\right]_{h+e_1h_1+(e_2-1)h_2+(e_3-1)h_3+(e_4-1)h_4}}{\left[1-(A_p)-n\right]_{h+e_1h_1+(e_2-1)h_2+(e_3-1)h_3+(e_4-1)h_4}}$$

$$\times \frac{\left[1-(D_s)-n\right]_{h+e_1h_1+e_2h_2+e_3h_3+e_4h_4}\left[(\alpha_{u_1})\right]_{h_1}\left[(\alpha_{u_2})\right]_{h_2}}{\left[1-(C_r)-n\right]_{h+e_1h_1+e_2h_2+e_3h_3+e_4h_4}\left[(\beta_{v_1})\right]_{h_1}\left[(\beta_{v_2})\right]_{h_2}}$$

$$\times \frac{\left[(\alpha_{u_3})\right]_{h_3}\left[(\alpha_{u_4})\right]_{h_4}\ (\sigma)_h\ \lambda^h\ (\lambda_1)_{h_1}\ \mu_1^{h_1}\mu_2^{h_2}\ \left(\mu_3 x_3^{e_3}\right)^{h_3}}{\left[(\beta_{v_3})\right]_{h_3}\left[(\beta_{v_4})\right]_{h_4}\ \xi^{\sigma+h}\ h!\ h_1!\ h_2!\ x_2^{e_1h_1+e_2h_2}\ h_3!}$$

$$\times \frac{\left(\mu_4 x_4^{e_4}\right)^{h_4}(-n+i)_{h+e_1h_1+e_2h_2+e_3h_3+e_4h_4}\,(-c-d-i-n-1)_{h+e_1h_1+e_2h_2+e_3h_3+e_4h_4}\,(-1)^{(p+q+r+s+1)h}}{h_4!\ \ (-n-c)_{h+e_1h_1+e_2h_2+e_3h_3+e_4h_4}\ \ \mu^{h+e_1h_1+e_2h_2+e_3h_3+e_4h_4}}$$

$$\times \frac{(-1)^{e_1(p+q+r+s+1)h_1}\,(-1)^{\{e_2(p+q+r+s+1)+p+q\}h_2}\,(-1)^{\{e_3(p+q+r+s+1)+p+q\}h_3}}{1}$$

$$\times (-1)^{\{e_4(p+q+r+s+1)+p+q\}h_4} \qquad \ldots (3.3)$$

where c is non-negative integer.

The single terminating factor $(-n+i)_{h+e_1h_1+e_2h_2+e_3h_3+e_4h_4}$ and makes all summations in (3.3) run up to □.

Corollary 1 : For $e_2 > 1$, $e_3 > 1$ and $e_4 > 1$, we have

$$M_n(x_1,x_2,x_3,x_4) = K\sum_{i=0}^{n} \frac{(n+c)!(2i++c+d+1)(-1)^i\,P_i^{(c,d)}\left(1+2x_2^e\right)}{(n-i)!(c+d+i+1)_{n+1}}$$

$$\times F^{2+q+s:u_1;u_2;u_3;u_4}_{p+r+1:v_1;v_2;v_3;v_4}\left[\begin{array}{l}\left[(-n+i):e,e_1,e_2,e_3,e_4\right],\\ \left[(-n-c):e,e_1,e_2,e_3,e_4\right],\end{array}\right.$$

$$\left[\begin{array}{ll}\left[(-c-d-i-n-1):e,e_1,e_2,e_3,e_4\right], & \left[(1-(B_q)-n):e,e_1,e_2-1,e_3-1,e_4-1\right],\\ \text{———————}, & \left[(1-(A_p)-n):e,e_1,e_2-1,e_3-1,e_4-1\right],\end{array}\right.$$

$$\begin{array}{l}\left[(1-(D_s)-n):e,e_1,e_2,e_3,e_4\right],\left[(\alpha_{u_1}):1\right],\left[(\alpha_{u_2}):1\right],\left[(\alpha_{u_3}):1\right],\left[(\alpha_{u_4}):4\right]\\ \left[(1-(C_r)-n):e,e_1,e_2,e_3,e_4\right],\left[(\beta_{v_1}):1\right],\left[(\beta_{v_2}):1\right],\left[(\beta_{v_3}):1\right],\left[(\beta_{v_4}):4\right]\end{array}$$

$$\begin{array}{l}[\sigma:1],[\lambda_1:1]\\ \text{——},\ \text{——}\end{array},\frac{\lambda(-1)^{e(p+q+r+s+1)}}{\mu\xi},\frac{\mu_1(-1)^{e_1(p+q+r+s+1)}}{(\mu x_2)^{e_1}},\frac{\mu_2(-1)^{e_2(p+q+r+s+1)+p+q}}{(\mu x_2)^{e_2}},$$

$$\left.\frac{\mu_3x_3^{e_3}(-1)^{e_3(p+q+r+s+1)+p+q}}{\mu^{e_3}},\frac{\mu_4x_4^{e_4}(-1)^{e_4(p+q+r+s+1)+p+q}}{\mu^{e_4}}\right] \qquad \ldots (3.4)$$

Corollary 2 : For $e_2 = 1$, $e_3 > 1$ and $e_4 > 1$, we get

$$M_n(x_1,x_2,x_3,x_4)=K\sum_{i=0}^{n}\frac{(n+c)!(2i++c+d+1)(-1)^i P_i^{(c,d)}(1+2x_1^e)}{(n-i)!(c+d+i+1)_{n+1}}$$

$$\times F^{2+q+s:u_1;u_2;u_3;u_4}_{p+r+1:v_1;v_2;v_3;v_4}\left[\begin{matrix}[(-n+i):e,e_1,1,e_3,e_4],\\ [(-n-c):e,e_1,1,e_3,e_4],\end{matrix}\right.$$

$$\left[\begin{matrix}[(-c-d-i-n-1):e,e_1,1,e_3,e_4],[(1-(B_q)-n):e,e_1,0,e_3-1,e_4-1],\\ \text{———————},\quad [(1-(A_p)-n):e,e_1,0,e_3-1,e_4-1],\end{matrix}\right.$$

$$[(1-(D_s)-n):e,e_1,1,e_3,e_4],[(\alpha_{u_1}):1],[(\alpha_{u_2}):1],[(\alpha_{u_3}):1],[(\alpha_{u_4}):4][\sigma:1][\lambda_1:1]$$

$$[(1-(C_r)-n):e,e_1,1,e_3,e_4],[(\beta_{v_1}):1],[(\beta_{v_2}):1],[(\beta_{v_3}):1],[(\beta_{v_4}):4]\quad —\quad —$$

$$\frac{\lambda(-1)^{(p+q+r+s+1)}}{\xi\,\mu},\frac{\mu_1(-1)^{e_1(p+q+r+s+1)}}{(\mu x_2)^{e_1}},\frac{\mu_2(-1)^{2(p+q)+r+s+1}}{\mu x_2},$$

$$\left.\frac{\mu_3 x_3^{e_3}(-1)^{e_3(p+q+r+s+1)+p+q}}{\mu^{e_3}},\frac{\mu_4 x_4^{e_4}(-1)^{e_4(p+q+r+s+1)+p+q}}{\mu^{e_4}}\right]\qquad \dots (3.5)$$

Corollary 3 : For $e_2 > 1$, $e_3 = 1$, $e_4 > 1$ we achieve

$$M_n(x_1,x_2,x_3,x_4)=K\sum_{i=0}^{n}\frac{(n+c)!(2i++c+d+1)(-1)^i P_i^{(c,d)}(1+2x_1^e)}{(n-i)!(c+d+i+1)_{n+1}}$$

$$\times F^{2+q+s:u_1;u_2;u_3;u_4}_{p+r+1:v_1;v_2;v_3;v_4}\left[\begin{matrix}[(-n+i):e,e_1,e_2,1,e_4],[(-c-d-i-n-1):e,e_1,e_2,1,e_4],\\ [(-n-c):e,e_1,e_2,1,e_4],\quad \text{———————},\end{matrix}\right.$$

$$\left[\begin{matrix}[(1-(B_q)-n):e,e_1,e_2-1,0,e_4-1],[(1-(D_s)-n):e,e_1,e_2,1,e_4],\\ [(1-(A_p)-n):e,e_1,e_2-1,0,e_4-1],[(1-(C_r)-n):e,e_1,e_2,1,e_4],\end{matrix}\right.$$

$$[(\alpha_{u_1}):1],[(\alpha_{u_2}):1],[(\alpha_{u_3}):1],[(\alpha_{u_4}):4][\sigma:1][\lambda_1:1]$$

$$[(\beta_{v_1}):1],[(\beta_{v_2}):1],[(\beta_{v_3}):1],[(\beta_{v_4}):4]\quad —\quad —$$

$$\frac{\lambda(-1)^{(p+q+r+s+1)}}{\xi\,\mu},\frac{\mu_1(-1)^{e(p+q+r+s+1)}}{(\mu x_2)^{e_1}},\frac{\mu_2(-1)^{e_2(p+q+r+s+1)+p+q}}{(\mu x_2)^{e_2}},$$

$$\left.\frac{\mu_3(-1)^{2(p+q)+r+s+1}x_3}{\mu},\frac{\mu_4 x_4^{e_4}(-1)^{e_4(p+q+r+s+1)+p+q}}{\mu^{e_4}}\right]\qquad \dots (3.6)$$

Corollary 4 : For $e_2 > 1$, $e_3 > 1$ and $e_4 = 1$, we get

$$M_n(x_1,x_2,x_3,x_4)=K\sum_{i=0}^{n}\frac{(n+c)!(2i++c+d+1)(-1)^i P_i^{(c,d)}(1+2x_1^e)}{(n-i)!(c+d+i+1)_{n+1}}$$

$$\times F^{2+q+s:u_1;u_2;u_3;u_4}_{p+r+1:v_1;v_2;v_3;v_4}\left[\begin{matrix}[(-n+i):e,e_1,e_2,e_3,1],[(-c-d-i-n-1):e,e_1,e_2,e_3,1],\\ [(-n-c):e,e_1,e_2,e_3,1],\quad \text{———————},\end{matrix}\right.$$

$$\left[\left[\left(1-\left(B_q\right)-n\right):e,e_1,e_2-1,e_3-1,0\right],\left[\left(1-\left(D_s\right)-n\right):e,e_1,e_2,e_3,1\right],\right.$$
$$\left[\left(1-\left(A_p\right)-n\right):e,e_1,e_2-1,e_3-1,0\right],\left[\left(1-\left(C_r\right)-n\right):e,e_1,e_2,e_3,1\right],$$
$$\left[\left(\alpha_{u_1}\right):1\right],\left[\left(\alpha_{u_2}\right):1\right],\left[\left(\alpha_{u_3}\right):1\right],\left[\left(\alpha_{u_4}\right):4\right][\sigma:1][\lambda_1:1]$$
$$\left[\left(\beta_{v_1}\right):1\right],\left[\left(\beta_{v_2}\right):1\right],\left[\left(\beta_{v_3}\right):1\right],\left[\left(\beta_{v_4}\right):4\right] \quad — \quad —$$
$$\frac{\lambda(-1)^{(p+q+r+s+1)}}{\mu\,\xi},\frac{\mu_1(-1)^{e_1(p+q+r+s+1)}}{(\mu x_2)^{e_1}},\frac{\mu_2(-1)^{e_2(p+q+r+s+1)+p+q}}{(\mu x_2)^{e_2}},$$
$$\left.\frac{\mu_3 x_3^{e_3}(-1)^{e_3(p+q+r+s+1)+p+q}}{\mu^{e_3}},\frac{\mu_4 x_4^{e_4}(-1)^{2(p+q)+r+s+1}}{\mu^{e_4}}\right] \quad \ldots (3.7)$$

Corollary 5 : For $e_2 = 1 = e_3 = e_4$, we get

$$M_n\left(x_1,x_2,x_3,x_4\right)=K\sum_{i=0}^{n}\frac{(n+c)!(2i++c+d+1)(-1)^i P_i^{(c,d)}\left(1+2x_1^e\right)}{(n-i)!(c+d+i+1)_{n+1}}$$
$$\times F_{p+r+1:v_1;v_2;v_3;v_4}^{2+q+s:u_1;u_2;u_3;u_4}\left[\begin{array}{l}\left[(-n+i):e,e_1,1,1,1\right],\left[(-c-d-i-n-1):e,e_1,1,1,1\right],\\ \left[(-n-c):e,e_1,1,1,1\right], \quad \text{———————},\end{array}\right.$$

$$\left[\left(\alpha_{u_1}\right):1\right],\left[\left(\alpha_{u_2}\right):1\right],\left[\left(\alpha_{u_3}\right):1\right],\left[\left(\alpha_{u_4}\right):4\right][\sigma:1][\lambda_1:1]$$
$$\left[\left(\beta_{v_1}\right):1\right],\left[\left(\beta_{v_2}\right):1\right],\left[\left(\beta_{v_3}\right):1\right],\left[\left(\beta_{v_4}\right):4\right] \quad — \quad —$$
$$\frac{\lambda(-1)^{(p+q+r+s+1)}}{\mu\,\xi},\frac{\mu_1(-1)^{e_1(p+q+r+s+1)}}{(\mu x_2)^{e_1}},\frac{\mu_2(-1)^{2(p+q)+r+s+1}}{(\mu x_2)},$$
$$\left.\frac{\mu_3 x_3(-1)^{2(p+q)+r+s+1}}{\mu},\frac{\mu_4 x_4(-1)^{2(p+q)+r+s+1}}{\mu}\right] \quad \ldots (3.8)$$

IV. Particular Cases :

1. On making the substitution $p = 0 = s = u_2 = \square_1 = p = 1 = r = v_2 = e = x_2 = \square\square = \square\square = \square$; $A_1 = 1$, $c_1 = \square = \square_1$, $e_2 = 2 = \square$; $\square_2 = -4$ and writting z for x_1 in (3.4), we get

$$R_{n,v}\left(\frac{1}{z}\right)=(v)_n\,2^n\sum_{i=0}^{n}\frac{(n+c)!(-1)^i(2i++c+d+1)P_i^{(c,d)}(1+2x)}{(n-i)!(c+d+i+1)_{n+1}}$$
$$\times F\left[\begin{array}{r}-n+i,-c-d-i-n-1;\\ -1\\ 1-v-n,-n,v-n-c;\end{array}\right]$$

where $R_{n,v}\left(\frac{1}{z}\right)$ are the Lommel Polynomials.

2. On setting $p = q = x_2 = \square_2 = \square_1$; $x_2 = 1 = e = \square\square = \square\square = s$; $r = \{1, 2\}$, $c_1 = \square$, $c_2 = \square$, $D_1 = \square\square + \square$; $e_2 = 2 = \square$; $\square_2 = -4$, and writting x for x_1 in (3.4), we get

$$G_n(\alpha,\beta;x)=\frac{(\alpha)_n(\beta)_n(2x)^n}{(\alpha+\beta)_n}\sum_{i=0}^{n}\frac{(n+c)!(-1)^i(2i++c+d+1)P_i^{(c,d)}(1+2x)}{(n-i)!(c+d+i+1)_{n+1}}$$

$$\times F\left[\begin{matrix}1-\alpha-\beta-n,-n+i,-c-d-i-n-1; \\ -1 \\ 1-\alpha-n,1-\beta-n,\nu-n-c;\end{matrix}\right]$$

where G_n(□, □, x) are Bedient Polynomials [4].

3. For $p = 0 = q = r = s$ = □$_1$ = h; $e = 1$ = □□= □□= □□= □□= $x_3 = e_3 = 3$; □$_3$ = –1 and 3y instead of x_1 in (3.4), we get

$$h_n^*(y)=\frac{(3y)^n}{1}\sum_{i=0}^{n}\frac{(n+c)!(-1)^i(2i++c+d+1)P_i^{(c,d)}(1+2x)}{(n-i)!(c+d+i+1)_{n+1}}$$

$$\times F\left[\begin{matrix}-n+i,-c-d-i-n-1; \\ \frac{1}{y^3} \\ -n-c;\end{matrix}\right]$$

where $h_n^*(y)$ are the humbert Polynomials.

4. If we take $q = r = 0 = s$ = □$_1$; $u_3 = v_3$; $e = 1 = p = x_3$ = □$_3$ = □□= □; □□= 2 = e_3 = 3; $A_1 = \frac{1}{2}$ and x for x_1 in (3.4), we get

$$g_n^*(x)=\frac{\left(\frac{1}{2}\right)_n(3x)^n}{1}\sum_{i=0}^{n}\frac{(n+c)!(-1)^i(2i++c+d+1)P_i^{(c,d)}(1+2x)}{(n-i)!(c+d+i+1)_{n+1}}$$

$$\times F\left[\begin{matrix}-n+c,-c-d-i-n-1; \\ \frac{(-1)^n}{4} \\ -n-c;\end{matrix}\right]$$

where $g_n^*(x)$ are the Pincherle Polynomials.

V. CONCLUSION:

In this article we have obtained many interesting new results for the quadruple hypergeometric polynomial set $M_n(x_1, x_2, x_3, x_4)$ followed by important and interesting particular cases. These are of at most important for scientist, engineers and physical scientist, because these occurs in the solution of integral equations, and analytic functions which diseribe physical problems.

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