Exponential Diophantine Equation $2^a + n^{2b} = c^2, n = 1,2,3,...$

[1]Janaki G, [2]Gowri Shankari A

[1]Associate Professor, [2]Assistant Professor
PG and Research Department of Mathematics
Cauvery College for Women (Autonomous), (Affiliated to Bharathidasan University), Trichy – 18, India

***Abstract*- In this work, we explore the Exponential Diophantine equation $2^a + n^{2b} = c^2, n = 1,2,3,...$, where *a, b,* and *c* are all non-negative integers, and we find that *(a, b, c) = (3, 0, 3)* is a unique non-negative integer solution to this equation.**

***Index Term*- Catalan's conjecture, Diophantine equation, integer solution.**

I. INTRODUCTION

Higher arithmetic, often known as number theory, is concerned with the characteristics of the natural numbers 1, 2, 3,.... These numbers must have ignited human interest from an early age, and there is evidence of some obsession with mathematics beyond the necessities of ordinary living in all ancient civilizations' records [1-4].

A Diophantine equation is one with the form $g(y_1, y_2,..., y_n) = 0$, where g is an n-variable function with $n \geq 2$. If f is an integral polynomial, then $g(y_1, y_2,..., y_n) = 0$ is an algebraic Diophantine equation. A solution to equation $g(y_1, y_2,..., y_n) = 0$ is an n-uple $(y_1^0 + y_2^0 + ... + y_n^0) \in Z_n$ that fulfil $g(y_1, y_2,..., y_n) = 0$. Solvable equations are those that have one or more solutions. Exponential Diophantine Equation $36^x + 3^y = z^2$ has two solutions (0,1,2) and (2,6,45), according to G. Janaki and P. Saranya [5]. Using the Catalan conjecture, G. Janaki and C. Saranya [6] demonstrated that there are non-negative integer solutions to the exponential Diophantine equation using jarasandha numbers. K. Kaleeswari, J. Kannan, and G. Narasimman [7] explored the Exponential Diophantine equations $3^x + 67^y = z^2$, $3^x + 127^y = z^2$ and found that (1,0,2) is the unique solution. Sudhanshu Agarwal and Lalit Mohan Upadhyaya [8] investigated Diophantine equations such as $783^x + 85^y = z^2$, which has a unique solution of (1,0,28). An exponential Diophantine equation is one that has integer exponents. Catalan conjecture solves several exponential Diophantine equations [9-14].

In this paper we consider the Diophantine equation $2^a + n^{2b} = c^2, n = 1,2,3,...$ where *a, b, c* are non-negative integers and determines that *(a, b, c)= (3, 0, 3)* is a unique non- negative integer solution of this equation.

II. PRELIMINARIES

Catalan's Conjecture:

The Diophantine equation $l^a - m^b = 1$, where *l, m, a, b* and *c* are integers such that $\min(l, m, a, b) > 1$, has a unique solution $(l, m, a, b) = (3,2,2,3)$

Lemma: 1

The Diophantine equation $2^a + 1 = c^2$, where *(a, c)* are non-negative integers, has a single solution $(a, c) = (3,3)$.

Proof:

If $(a, c) \in N \cup \{0\}$. If $a = 0$, then $c^2 = 2$ is non – viable. so assume $a \geq 1$.

Then $c^2 - 1 = 2^a = 2^{a-u}2^u \Rightarrow (c+1) = 2^{a-u}, (c-1) = 2^u$ (1)

From (1), $2 = 2^{a-u} - 2^u \Rightarrow u = 1$ then $2^{a-u} = 2 \Rightarrow a = 3.$ (2)

Apply (2) in $2^a + 1 = c^2$, one can get $c = 3$.

$\therefore (a, c) = (3,3).$

Lemma: 2

The Diophantine equation $1 + n^{2b} = c^2, n = 1,2,3,...$ where *(b, c)* are non-negative integers, has no integer solution.

Proof:

If $(b, c) \in N \cup \{0\}$. If $b = 0$, then $c^2 = 2$ is non – viable. So assume $b \geq 1$.

Then $c^2 - 1 = n^{2b} = n^{2b-u}n^u \Rightarrow (c+1) = n^{2b-u}, (c-1) = n^u$ (3)

From (3), $2 = n^{2b-u} - n^u \Rightarrow 1.2 = n^u(n^{2b-2u} - 1) \Rightarrow u = 0$ since $n = 1,2,3,...$ (4)

Then (4) becomes $\left(n^{2b}-1\right)=2 \Rightarrow n^{2b}=3$ not possible for any $n=1,2,3,...$

$\therefore$ The Diophantine equation $1+n^{2b}=c^2, n=1,2,3,...$ where (b, c) are non-negative integers, has no integer solution.

III. MAIN RESULTS

Theorem:

(a, b, c)= (3, 0, 3) is a unique non- negative integer solution to the Diophantine equation $2^a+n^{2b}=c^2, n=1,2,3,...$ where *a*, *b*, and *c* are all non-negative integers.

Proof:

Assume *a*, *b*, and *c* are all non-negative integers such that $2^a+n^{2b}=c^2, n=1,2,3,...$ (5)

By Lemma 1, we have $a \geq 1$.

Then $2^a+n^{2b}=c^2$ as $2^a+\left(n^b\right)^2=c^2 \Rightarrow c^2-\left(n^b\right)^2=2^a=2^{a-u}2^a$ (6)

From (6), $(c+n^b)=2^{a-u}, (c-n^b)=2^u$ (7)

Using (7) one may get, $2(n^b)=2^{a-u}-2^u \Rightarrow 2(n^b)=2^u\left(2^{a-2u}-1\right) \Rightarrow u=1$ then $2^a-1=n^b$.

Suppose $b=0$ then $a=1$ then (5) becomes $3=c^2$ which is not possible.

Hence Diophantine equation $2^a+n^{2b}=c^2, n=1,2,3,...$ where *a*, *b*, and *c* are all non-negative integers has unique non-negative integer solution (3, 0, 3).

Some Numerical Solutions:

n	Diophantine equation $2^a+n^{2b}=c^2$	Solution
1	$2^a+1^b=c^2$	(3,0,3)
2	$2^a+4^b=c^2$	(3,0,3)
3	$2^a+9^b=c^2$	(3,0,3)
4	$2^a+16^b=c^2$	(3,0,3)
5	$2^a+25^b=c^2$	(3,0,3)

IV. CONCLUSION

In this study, we looked for non-negative integer solutions to the Diophantine equation $2^a+n^{2b}=c^2, n=1,2,3,...$, where *a*, *b*, and *c* are all non-negative integers, and demonstrated that *(a, b, c) = (3, 0, 3)* is a unique non-negative integer solution to this Diophantine equation.

In future, the approach described in this study can be used to solve any other Diophantine equations and related systems.

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