A new prospect of Matroid Theory

[1]Mr. Ajay Kumar, [2]Dr. Sushil Kumar Jamariar, [3]Mr. Ratan Priye, [4]Mr. Subodh Kumar Paswan

[1]Assistant Professor of Mathematics, [2]Associate Professor & H.O.D., [3,4]Student of M.Sc. (Mathematics)
Department of Mathematics
Dr. C.V. RAMAN UNIVERSITY
Bhagwanpur, Vaishali, Bihar.

***Abstract*- A Matroid is a structure that generalizes the properties of independence . The name 'Matroid ' suggests a structure related to a matrix and indeed , Matroids were introduced by Whiteney in 1938 to provide a unifying abstract treatment of dependence in linear algebra and graph theory ; Relevant applications are found in graph theory and linear Algebra. There are several ways to define a Matroid, each related to the concept of independence. A Characteristic of matroid is that they can be defined on many different be it equivalent ways . This paper will focus on the definitions of matrvids in terms of independent sets, bases, the rank function and cycles. This paper consists of preliminaries and each of the rest casists of a particular definition of Matroid and its application in graph theory and linear Algebra. Here we observe how both graphs and Matroids can be viewed as Matrices.**

***Keyword:* Graph theory, Linear Algebra, Bases, Rank Function , Circuite, Vertex-edge, Incidence Matrix.**

Introduction

We first familiar with the concept of a graph and then begin to in copration graphs in to Matroids theory.

Definition 1:-

A Simple graph G is a non empty set of finite set of disordered pairs of elements called edge [4]

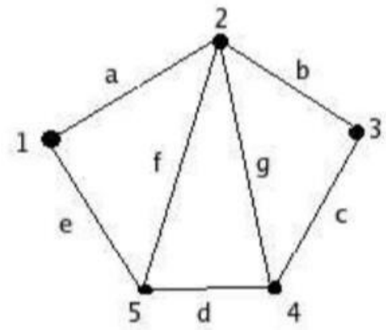

fig-1 Graph G

In figure 1 , the set of vertices V (G) are a {1,2,3,4,5} and the set of edges E(G) are {p,q,r,s,t,u,v}. Matroids focus on the set of edges, E(G) , as the elements of a matroid.

Definition 2 :-

Given the distinct vertices p_0 , p_1 , p_2 ……………..p_m , a path is closed if $p_0 = p_m$. A closed path is also known as cycle in graph theory [4]

Definition 3 :-

A connected graph having no cycles is a tree while the union of trees is a forest . Clearly graph is a forest. Iff it has no cycles.

Definition 4 :-

A spannig tree of a connected graph G is a subgraph T of G such that T is a tree and V(T) = V(G) [4]

Basic Linear Algebra

A is a 5x8 matrix and its column vectors are R^5 . The set of column vectors of the matrix A are {a,b,c,d,e,f,g,h} we will focus on the set of column vectors in a matrix as Two elements of a matroid.

$$A = \begin{pmatrix} a & b & c & d & e & f & g & h \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Bases

Definition 5

A Matroid M consist of a non-empty finite set E and a non-empty cellectorn P A subset of E , called base satisfying the following proportion [1]

P1 No base properly contains another base

P2 if P1 and P2 are base and if {e} is any element of **P1** , then there is an element F of P2 such that P1 – {e} U {f} is also a base.

P2 is known as the exchange property. This property state that if an element is removed from P1 then there exists an element is formed when that element is added to P(i) , we can use this priporla P(ii) to show that every base in a matroid has the some number of element

Theorem 1 Every base of a matroid has the equal and same of element

Proof : Let us suppose that the two base of a matorid M, P_1 and P_2 contain . P_2 such that a new base P_3 is formed By observing the set of bases losted above. We can see that P(i)is satisfied because no base property contains another base . We can now demonstrate P(ii) by using this property with two bases . if we choose P_1 = {p,q,r,s} and P_2 = {r,u,p,t} then we can see the spanning tree of P_1 and P_2 in fig 2 and 3 different number of elements Such that $|p_1| < |p_2|$. Now assume that there is equal and same element , {e} ∈ M , such that $e_1 \in p_1$, but $e_1 \notin p_2$. If we diminish $\{e_1\}$ from p_1 , then by p(ii), we know there $e_2 \in p_2$, but $e_2 \notin p_1$ such that $p_3 = p_1 / \{e_1\}$, ∪ $\{ e_2 \}$. Where p_3 is a base in M. There fore $|p_1| = |p_2|$ but that $|p_2| \neq |p_1| = |p_3|$

Now we continue the process of exchanging elements defined by the property p(ii) , n number of times then there will be no element initially in p_1 that is not in the base p_n. Therefore for all $e \in p_n$ the element e is also in p_2 and thus $p_n \subseteq p_2$. From p(i) we know that no base property contains another base . This is a contradiction.

There fore every base has two equal and same numbers of elements.

Definition 6

Let us suppose G be a graph with n vertices . A spanning tree is a connected sub graph that uses all vertices of G that has n-1 edges [4]

If we refer back to fig 1 the base of the graph G are {p,q,r,s}, {p,t,s,r}, {q,r,s,t}, {q,p,t,s}, {r,q,p,t},{r,q,u,t}, {r,s,u,p},{r,v,p,t}, {r,v,u,t}

An example in graph theorems

Let us take a base of our matroid to be a spanning tree of G. Here given a definition of Spanning tree

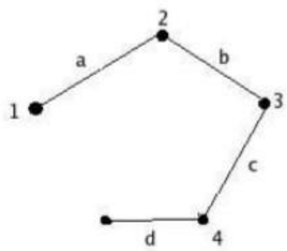

Fig 2. The Spanning Tree P1

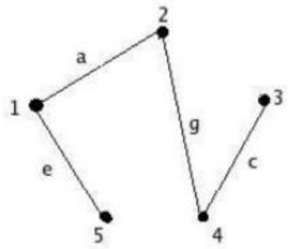

Fig 3. The Spanning Tree P2

Here we have to note that each spanning tree has 5 vertices and 4 edge .we can demonstrate P(ii) by diminishing an element {P} form P1, and then there exists an element in P_2 such that a new base is created P3 = P1 /{P}U{t}. fig 4 shows the new base P_3

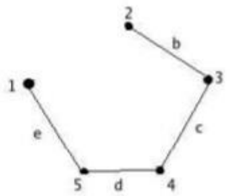

Fig- 4 The spanning Tree P3

A similar computation work for any chace of base Because we take the spanning trees of a graph to be the based of a matroids , we can canclude that the base of a matroid have the same number of elements and by the definition of a spanning has n-1 element .(if there are tree n vertices).

Independent sets in graph theory

Let us take independent sets of a graph to be sets of edges in a graph that do not contain a cycle . In graph theory , a cycle is a closed path . Another definition of independent set in graph theory uses forests which is defined as follows?

Definition 7

A forest is a graph that do not contains cycle . A connected forest is a tree.

We may say that the independent set of a graph are the edge set of the forests contained in the graph Fig – 5

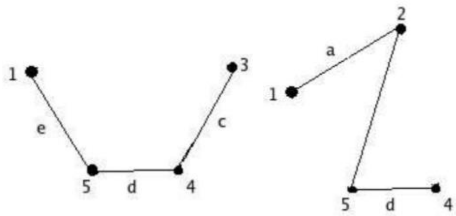

Fig- 6- An example of a forest in G

Vertex – edge incidence matrix

Now we are going to link graph theory and linear algebra by translating a graph to a unique matrix and vice versa .Using the language of matroids to motivate our discussion.The vertex edge inerdence matrix demonstrate the relation ship between a matrix and a graph . The following is a formed definition of a vertex edge incidence matrix given by james oxley.

Theorem Let G be a graph and AG be its vertex- edge incidence matrix, then the entries of AG are vlewed modulo (2) , its vector matroids M[AG] has as its independent sets all of E(G) that do not contain the edge of a cycle then M[AG] = M[G] and every graph matroid is binary[3]

The idea of a matrix being viewd mod (2) means that the enteries of the matrix are either 0 or 1 .

Now the example of a vertex edge incidence matrix using the graph in fig 9.

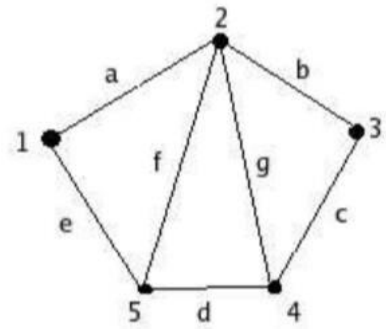

Fig 7

If an edge and a vertex are incident on a graph the corresponding entry in the matrix is 1. Otherwise , if an edge and vertex are not incident then the corresponding entry in the matrix is 0.

$$\begin{pmatrix} p & q & r & s & t & u & v \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We may take a relationship between graphs and matrices . in the vertex – edge incidence matrix if we may see that the rank of the set {a,e,f} is 2 because any subset containg two elements does not contain a cycle. In thegraph in fig 9 . We can see that the set {a,e,f} is a cycle the sum of the column vectore corresponding to the set of edges in our example is

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We may also see that this set of vectors is minimally dependent . If we take any one vector from the set the become an independent set.
The rank of the column vectors corresponding to the set {a,e,f } is also two , because any subset of column vectors , containing two elements does not contain a cycle.
One base of G is the set of edges {a,b,c,d} .

The corresponding vectors are

$$N = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

We have see that the set of vectors are maximal independent set , because
$|N| = r(N) = 4$.
Therefor , the set of vectors , N is also a base.
Here link between graph theory and linear algebra by using of matroids to motivate our discussion and generalize the properties of independence.

Conclusion

We find that Matroid arises naturally in Combinatorial optimization and can be used as a frame work for approaching a diverse Variety of Combinatorial problems. We have discussed definition of matroids and its application in graph theory and Linear Algebra we have got how both graphs and matroids can be viewed as matroids and also linked graph theory and Linear algebra by transtating a graph to a unique matrix and vice versa.

Acknowledgement

The authors are Thankful to Prof (Dr.) Basant Singh, Provice Chancellor,Dr. C.V. Raman University , Vaishali Bihar, and Prof (Dr.) Dharmendra Kumar Singh, Dean Academic , Dr. C. V. Raman University, Vaishali ,Bihar, India . Thanks to library and its incharge , Dr. C .V . Raman University ,Vaishali, Bihar, India. For extending all facilities in the completion of the present research work

REFERENCES:

1. Bender, Edward A, Willium S Gill (2010) lists, Decisions and Graphs.With an introduction to Probability.
2. Bollobas, Bela, Riordan OM (2003)Mathematical , result on Scale free random graph in Hand book of Graph and Networks ‘ CS. Bornhold t and HG Schuster (eds)] 1sted, weinheimwilemvelt.
3. Mahadev NUR PeledUriN (1995) threshold graph and Related topics North Holland.
4. New man Mark (2010) Networks An Introduction (Oxford university Press)
5. Kepner, Jeremy, Gilbert John (2011) Graph A lgorithms in the language of Linear Algebra. Philadelphia, Penh Sylvannia SIAM ISBN978-0-898719
6. H HILLMAN . MATROID theory.
7. DC Lay . SR Lay And J . MC Donald Linear Algebra and its application 2016
8. J G Oxley Matroid theory Volume 3 Oxford university Press.
9. RJ Wilson introducation to Sraph theory 1979 Pearson education india.