A STUDY ON L-FUZZY VECTOR SUBSPACES AND ITS FUZZY DIMENSION

R. Geetha

Assistant Professor
Department of Mathematics,
KG College of Arts and Science, Coimbatore-641035, Tamil Nadu, India.

ABSTRACT: This paper gives the definition of L-fuzzy vector subspace and defining its dimension by an L-fuzzy natural number. It is proved that for a finite dimensional L-fuzzy vector subspace, the intersection of two L-fuzzy vector subspace is also a L-fuzzy vector subspace and also the inequality $\dim(\widetilde{E1}+\widetilde{E2})+\dim(\widetilde{E1}\cap\widetilde{E2})=\dim\widetilde{E1}+\dim\widetilde{E2}$ holds without any restricted conditions.

INTRODUCTION

Fuzzy vector space was introduced by Katsaras and Liu. The dimension of a fuzzy vector space is defined as a n-tuple by Lowen. The study of fuzzy vector spaces started as earlyas 1977. A fuzzy subset of a non-empty set S is a function from S into [0,1]. Let A denote a fuzzy subspace of V over a fuzzy subfield Kof F and let X denote a fuzzy subset of V such that $X \subseteq A$. Let$\langle X\rangle$ denote the intersection of all fuzzy subspaces of V over Kthat contain X and are contained in A.

PRELIMINARIES

Consider the set X and completely distributive lattice L. Let the power set of X be 2^x and the set of all L-fuzzy sets on X be L^x respectively.For any $A \subseteq X$,the cardinality of A be denoted by |A|.An element L is called a prime element if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$ and an element L is called co-prime if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$.

The set of non-unit prime elements in L is denoted by P(L) and the set of non-zero co-prime elements in L is denoted by J(L). The binary relation < is defined by for all $a,b \in L$, $a < b$ if and only if for every subset $D \subseteq L$ with $a \leq d$,the relation $b \leq \sup D$ is possible only when $d \in D$ with $a \leq d$.The greatest minimal family of b is denoted by $\beta(b) = \{ a \in L: a < b\}$ and $\beta^*(b) = \beta(b)\cap J(L)$.Moreover for $b \in L$ we define $\alpha(b) = \{ a \in L: a <^{op} b\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$. In a completely distributive lattice L, there exist $\alpha(b)$ and $\beta(b)$ for each $b \in L$ and $b= \vee \beta(b) = \wedge \alpha(b)$.

Let $\mathbb{N}(L)$ denotes the L-fuzzy natural number and the relation of α-cut sets are defined as follows

For any $\lambda, \mu \in \mathbb{N}(L)$, $a \in L$,

(i) $(\lambda+\mu)_{(a)} \subseteq \lambda_{(a)} + \mu_{(a)} \subseteq \lambda_{[a]}+\mu_{[a]} \subseteq (\lambda+\mu)_{[a]}$; (ii) $(\lambda+\mu)^{(a)} \subseteq \lambda^{(a)}+\mu^{(a)} \subseteq \lambda^{[a]}+\mu^{[a]} \subseteq (\lambda+\mu)^{[a]}$;

(iii) For any $\lambda, \mu \in \mathbb{N}(L)$ and $a \in P(L)$ implies $(\lambda+\mu)^{(a)} = \lambda^{(a)}+\mu^{(a)}$.

1. L-FUZZY VECTOR SUBSPACES

DEFINITION 1.1

L-FUZZY VECTOR SUBSPACE

L-Fuzzy Vector Subspace (LFVS) is a pair $\tilde{E}$= (E, μ) where E is a vector space on field F, $\mu:E\rightarrow L$ is a map with the property that for any $x,y \in E$ and $k,l \in F$ such that $\mu(kx+ly) \geq \mu(x)\wedge \mu(x)$.

When L= [0,1] then L-Fuzzy Vector Subspace becomes fuzzy vector subspace.Let $\tilde{E}$= (E, μ) be a member of LFVS then

$\tilde{E}_{[a]}= \mu_{[a]} =\{x \in E: \mu(x) \geq a\}$, $\tilde{E}_{(a)}= \mu_{(a)} =\{x \in E: a \in \beta(\mu(x))\}$.

$\tilde{E}^{[a]}= \mu^{[a]} =\{x \in E: a \notin \alpha(\mu(x))\}$, $\tilde{E}^{(a)}= \mu^{(a)} =\{x \in E: \mu(x) \not\leq a\}$.

THEOREM: 1.1

Let E be a vector space, μ ∈ L^E and $\tilde{E}$= (E, μ) then the following statements are equivalent.

(i) $\tilde{E}$ is an L-fuzzy vector subspace.(ii) For all a ∈ L, $\tilde{E}_{[a]}$ is a vector space.

(iii) For all a ∈ J(L), $\tilde{E}_{[a]}$ is a vector space. (iv) For all a ∈L, $\tilde{E}^{[a]}$ is a vector space.

(v) For all a ∈ P(L), $\tilde{E}^{[a]}$ is a vector space. (vi) For all a ∈ P(L), $\tilde{E}^{(a)}$ is a vector space.

PROOF:

It is enough if we prove 1⟺4 and 1⟺6

(i) Assume that $\tilde{E}$ is an L-fuzzy vector subspace

Suppose that x,y ∈ $\tilde{E}^{[a]}$ then a ∉ α(μ (x)) and a ∉ α(μ (y))

i.e. a ∉ α(μ (x)) ∪ α(μ (y)) = α(μ(x)Λ μ(y))

then α(μ(x)Λ μ(y)) ⊇ α(μ(kx+ly))

We have a ∉ α(μ(kx+ly))

Hence kx+ly ∈ $\tilde{E}^{[a]}$

Therefore $\tilde{E}^{[a]}$ is a vector space.

Suppose that for all a ∈ L, $\tilde{E}^{[a]}$ is a vector space.

Let x,y € E and k,l ∈F then kx+ly ∈ $\tilde{E}^{[a]}$ if and only if x ∈ $\tilde{E}^{[a]}$ and y ∈ $\tilde{E}^{[a]}$

We have $\mu(kx+ly) = \bigwedge_{a\in L} (a \wedge \tilde{E}^{[a]})(kx+ly)$

$= \bigwedge_{a\in L} (aV(\tilde{E}^{[a]}(x) \wedge \tilde{E}^{[a]}(y))$

$= (\bigwedge_{a\in L} (aV(\tilde{E}^{[a]}(x))) \wedge (aV(\tilde{E}^{[a]}(x))$

$= \mu(x) \wedge \mu(y)$

Therefore $\tilde{E}$ is an L-fuzzy vector subspace.

Hence 1⟺4

(ii) Suppose that x,y ∈ $\tilde{E}^{(a)}$ then μ(x) ≰ a and μ(y) ≰a

Since a ∈ P(L) then μ(x)Λ μ(y) ≰ a (Since $\tilde{E}$=(E,μ) is an LFVS)

That is μ (kx+ly) ≰ a

Implies kx+ly ∈ $\tilde{E}^{(a)}$

Therefore $\tilde{E}^{(a)}$ is a vector space.

Assume x, y ∈ E and k, l ∈ F then

kx+ly ∈ $\tilde{E}^{(a)}$ if and only if x ∈ $\tilde{E}^{(a)}$ and y ∈ $\tilde{E}^{(a)}$ (Since $\tilde{E}^{(a)}$ is a vector space)

We have $\mu(kx+ly) = \bigwedge_{a\in P(L)} (aV\tilde{E}^{(a)})(kx+ly)$

$$= \bigwedge_{a\in P(L)} (a \vee \tilde{E}^{(a)}(x) \wedge \tilde{E}^{(a)}(y)))$$

$$= \left(\bigwedge_{a\in P(L)} (a \vee \tilde{E}^{(a)}(x))\right) \wedge \left(\bigwedge_{a\in P(L)} (a \vee \tilde{E}^{(a)}(y))\right)$$

$$= \mu(x) \wedge \mu(y)$$

Therefore $\tilde{E}$ is an L-fuzzy vector subspace. Therefore 1⟺6

Hence the Theorem.

THEOREM: 1.2

Let V be a vector space, μ:E→L is a map and for all a,b ∈ L,β (a∧b) = β(a) ∩ β(b) then the following statements are equivalent:

(1) $\tilde{E}$ is an L-fuzzy vector subspace. (2) For all a ∈ L, $\tilde{E}_{(a)}$ is a vector space.

PROOF:

Assume $\tilde{E}$ is an L-fuzzy vector subspace.

Suppose that x,y ∈ $\tilde{E}_{(a)}$ then a ∈ $\beta(\mu(x))$ and a ∈ $\beta(\mu(y))$

i.e a ∈ $\beta(\mu(x)) \cap \beta(\mu(y))$

Since for all a,b ∈ L, $\beta(a \wedge b)= \beta(a) \cap \beta(b)$ and $\tilde{E}$ is an L-fuzzy vector subspace

i.e a ∈ $\beta(\mu(x) \wedge \mu(y)) \subseteq \beta(\mu(ax+by))$

⇒ ax+by ∈$\tilde{E}^{(a)}$

Therefore $\tilde{E}_{(a)}$ is a vector space.

Next assume that for all a ∈ L, $\tilde{E}_{(a)}$ is a vector space .

Let x,y ∈ E and k,l ∈ F then kx+ly ∈ $\tilde{E}_{(a)}$ if and only if x ∈ $\tilde{E}_{(a)}$ and y ∈ $\tilde{E}_{(a)}$ (Since $\tilde{E}_{(a)}$ is a vector space)

We have μ (kx+ly) $= \bigvee_{a\in L} (a \wedge \tilde{E}_{(a)})(kx+ly)$

$$= \bigvee_{a\in L} (a \wedge (\tilde{E}_{(a)}(x) \wedge \tilde{E}^{(a)}(y)))$$

$$= \left(\bigvee_{a\in L} (a \wedge (\tilde{E}_{(a)}(x))\right) \wedge \left(\bigvee_{a\in L} (a \wedge (\tilde{E}_{(a)}(y))\right)$$

$$= \mu(x) \wedge \mu(y)$$

Therefore $\tilde{E}$ is an L-fuzzy vector subspace.

Therefore the above two statements are equivalent.

DEFINITION 1.2

Let $\widetilde{E_1}$=(E,μ_1) and$\widetilde{E_2}$ =(E,μ_2) be two fuzzy vector subspaces on E. The intersection of $\widetilde{E_1}$ and $\widetilde{E_2}$is defined as

$\widetilde{E_1} \cap \widetilde{E_2}$ = (E, $\mu_1 \wedge \mu_2$) and the sum of $\widetilde{E_1}$ and $\widetilde{E_2}$ is defined as $\widetilde{E_1} + \widetilde{E_2}$ = (E, $\mu_1+\mu_2$)

Where $\mu_1+\mu_2$ is defined as for all x ∈ E , $(\mu_1+\mu_2)(x) = \bigvee (\mu_1(x_1) \wedge \mu_2(x_2))$

$$x = x_1 + x_2$$

$$= \bigvee_{x_1 \in E} (\mu_1(x_1) \wedge \mu_2(x - x_1)).$$

DEFINITION 1.3

Let $\widetilde{E_1}$=(E,μ_1) and $\widetilde{E_2}$=(E,μ_2) be two members on LFVS and E=$E_1 \oplus E_2$ be the direct sum of $\widetilde{E_1}$ and $\widetilde{E_2}$ defined as $E_1 \oplus E_2$ =(E, $\mu_1 \oplus \mu_2$) where $\mu_1 \oplus \mu_2$ is defined as for all x ∈ E, x = $x_1 \oplus x_2$, $x_i \in E_i$, i=1,2

$$(\mu_1 \oplus \mu_2)(x) = (\mu_1 \oplus \mu_2)(x_1 \oplus x_2) = \mu_1(x_1) \wedge \mu_2(x_2).$$

THEOREM: 1.3

Let $\widetilde{E_1}$=(E,μ_1) and $\widetilde{E_2}$=(E,μ_2) be two members on LFVS on E we have

(i) $\widetilde{E_1} \cap \widetilde{E_2}$ is a member of LFVS on E. (ii) $\widetilde{E_1} + \widetilde{E_2}$ is a member of LFVS on E.

PROOF:

Given $\widetilde{E_1}$ and $\widetilde{E_2}$ be two members on LFVS then $\mu_1(kx+ly) \geq \mu_1(x) \wedge \mu_1(y)$ and $\mu_2(kx+ly) \geq \mu_2(x) \wedge \mu_2(y)$

To prove $\widetilde{E_1} \cap \widetilde{E_2}$ is a member of LFVS on E

$\widetilde{E_1} \cap \widetilde{E_2}$= (E, $\mu_1 \wedge \mu_2$)   (By definition 3.2)

Consider (E, $\mu_1 \wedge \mu_2$) = $\mu_1 \wedge \mu_2(kx+ly)$

$= \mu_1(kx+ly) \wedge \mu_2(kx+ly)$

$\geq (\mu_1(x) \wedge \mu_1(y) \wedge \mu_2(x) \wedge \mu_2(y))$

Therefore $\widetilde{E_1} \cap \widetilde{E_2}$ is a member of LFVS on E.

Similarly we can prove $\widetilde{E_1} + \widetilde{E_2}$ is also a member of LFVS on E.

THEOREM: 1.4

Let $\widetilde{E_1}$=(E,μ_1) and $\widetilde{E_2}$=(E,μ_2) be two members on LFVS on E .Suppose that for any a,b ∈ L, we have

β (a∧b) = β(a) ∩ β(b) then (1) $(\widetilde{E_1} \cap \widetilde{E_2})_{(a)} = (\widetilde{E_1})_{(a)} \cap (\widetilde{E_2})_{(a)}$ (2) $(\widetilde{E_1} \cap \widetilde{E_2})_{(a)} = (\widetilde{E_1})_{(a)} \cap (\widetilde{E_2})_{(a)}$.

2.FUZZY DIMENSION OF L-FUZY VECTOR SUBSPACES

DEFINITION 2.1

Let ℕ(L) be the family of L-fuzzy natural number. The map dim:LFVS⟶ ℕ(L) is

defined by $\dim \tilde{E}(n) = \bigvee_{a \in L} (a \wedge \dim \tilde{E}_{[a]})_{(n)}$

is called the L-fuzzy dimensional function of the L-fuzzy vector subspace $\tilde{E}$,it is an fuzzy natural number.

Also $\dim \tilde{E}(n) = \vee \{a \in L: \dim \tilde{E}_{[a]} \geq n\}$.

THEOREM 2.1

Let $\widetilde{E1}$=(E,μ_1) and $\widetilde{E2}$=(E,μ_2) be two L-fuzzy vector subspaces then the following equalities holds

$$\dim(\widetilde{E1} + \widetilde{E2}) + \dim(\widetilde{E1} \cap \widetilde{E2}) = \dim \widetilde{E1} + \dim \widetilde{E2}$$

PROOF:

Given $\widetilde{E1}$ and $\widetilde{E2}$ be two L-fuzzy vector subspaces then the sum of $\widetilde{E1}$ and $\widetilde{E2}$ be denoted by $\widetilde{E_1} + \widetilde{E_2}$

$$(\dim(\widetilde{E1} + \widetilde{E2}) + \dim(\widetilde{E1} \cap \widetilde{E2}))^{(a)} = (\dim(\widetilde{E1} + \widetilde{E2}))^{(a)} + (\dim(\widetilde{E1} \cap \widetilde{E2}))^{(a)}$$

$$= \dim(\widetilde{E1} + \widetilde{E2})^{(a)} + (\dim(\widetilde{E1} \cap \widetilde{E2})^{(a)}$$

$$= \dim(\widetilde{E1}^{(a)} + \widetilde{E2}^{(a)}) + \dim(\widetilde{E1}^{(a)} \cap \widetilde{E2}^{(a)})$$

$$= \dim\widetilde{E1}^{(a)} + \dim\widetilde{E2}^{(a)}) - \dim(\widetilde{E1}^{(a)} \cap \widetilde{E2}^{(a)}) + \dim(\widetilde{E1}^{(a)} \cap \widetilde{E2}^{(a)})$$

$$= \dim\widetilde{E1}^{(a)} + \dim\widetilde{E2}^{(a)}$$

Therefore $\dim(\widetilde{E1} + \widetilde{E2}) + \dim(\widetilde{E1} \cap \widetilde{E2}) = \dim\widetilde{E1} + \dim\widetilde{E2}$

Hence the theorem.

CONCLUSION

In this paper L-fuzzy vector subspace is defined and showed that its dimension is an L-fuzzy natural number. Based on the definitions some properties of crisp vector space s are hold in finite dimensional vector spaces. In particular the equality dim $(\widetilde{E1} + \widetilde{E2}) + \dim(\widetilde{E1} \cap \widetilde{E2}) = \dim\widetilde{E1} + \dim\widetilde{E2}$ holds without any restricted conditions.

REFERENCE

[1] Katsaras, A.K. and Liu, D.B. (1977) Fuzzy Vector Spaces and Fuzzy Topological VectorSpaces. Journal of Mathematical Analysis and Applications , 58, 135-146.
[2] Lubczonok, G. and Murali, V. (2002) On Flags and Fuzzy Subspaces of Vector Spaces.Fuzzy Sets and Systems , 125, 201-207.
[3] Abdukhalikov, K.S., Tulenbaev, M.S. and Umirbarv, U.U. (1994) On Fuzzy Bases of VectorSpaces. Fuzzy Sets and Systems , 63, 201-206.
[4] Abdukhalikov, K.S. (1996) The Dual of a Fuzzy Subspace. Fuzzy Sets and Systems , 82, 375-381.
[5] Lubczonok, P. (1990) Fuzzy Vector Spaces. Fuzzy Sets and Systems , 38, 329-343.
[6] Lowen, R. (1980) Convex Fuzzy Sets. Fuzzy Sets and Systems , 3, 291-310.
[7] Shi, F.G. and Huang, C.E. (2010) Fuzzy Bases and the Fuzzy Dimension of Fuzzy VectorSpaces. Mathematical Communications , 15, 303-310.
[8] Gierz, G., et al . (2003) Continuous Lattices and Domains. Encyclopedia of Mathematics andits Applications, Cambridge University Press, Cambridge.
[9] Dwinger, P. (1982) Characterizations of the Complete Homomorphic Images of a CompletelyDistributive Complete Lattice I. Indagationes Mathematicae (Proceedings), 85, 403-414.
[10] Wang, G.-J. (1992) Theory of Topological Molecular Lattices. Fuzzy Sets and Systems , 47,351-376.
[11] Huang, H.-L. and Shi, F.-G. (2008) L-Fuzzy Numbers and Their Properties. InformationSciences , 178, 1141-1151.
[12] Shi, F.-G. (2000) L-Fuzzy Relation and L-Fuzzy Subgroup. Journal of Fuzzy Mathematics , 8,491-499.
[13] Negoita, C.V. and Ralescu, D.A. (1975) Applications of Fuzzy Sets to Systems Analysis, InterdisciplinarySystems Research Series 11, Birkhaeuser, Basel.
[14] Shi, F.-G. (1995) Theory of Lβ-Nested Sets and Lα-Nested Sets and Its Applications. Fuzzy Systems and Mathematics , 4, 65-72. (In Chinese)
[15] Shi, F.-G. (1996) L-Fuzzy Sets and Prime Element Nested Sets. Journal of MathematicalResearch and Exposition, 16, 398-402. (In Chinese)
[16] Shi, F.-G. (1996) Theory of Molecular Nested Sets and Its Applications. Journal of YantaiTeachers University , 1, 33-36. (In Chinese)
[17] Shi, F.-G. (2009) A New Approach to the Fuzzification of Matroids. Fuzzy Sets and Sys