APPLICATION OF DIFFERENTIAL EQUATION IN SIMPLE HARMONIC MOTION

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***Abstract*: This study has been undertaken to investigate the detailed explanation of simple harmonic motion and to obtain the expression of velocity, acceleration, amplitude, frequency and the position of the particle executing the motion and its application in clocks, musical instruments, bungee jumping etc.,**

***Keywords*: Restoring, force, vibration, force, oscillation, projection, displacement, energy, sound, string.**

INTRODUCTION

Simple Harmonic Motion (SHM) is a type of periodic motion or oscillation motion where restoring force is proportional to displacement and direction that is opposite to displacement. When an object moves to and fro along the fixed line then the motion is said to be simple harmonic function.

Simple Harmonic Motion serves as mathematical model for various motions such as oscillation of spring, pendulum and molecular vibrations. Simple harmonic motion is the basis for complicated motions through Fourier analysis. The linear elastic restoring force in simple harmonic motion is given by Hooke's law.

SIMPLE HARMONIC MOTION

The movement of the object back and forth through equilibrium or centre position so maximum displacement on one side is equal to maximum displacement on another side. The time taken for completion of each vibration is same and force is responsible for the motion directed towards the equilibrium.

PRELIMINARIES

This section deals with some fundamental definitions of Simple harmonic motion.

ACCELERATION

The differential equation of Simple harmonic motion is given in the form,

$$\frac{d^2x}{dt^2} + (k/m)\,x = 0$$

Where m is mass of particle
K is force constant
X is displacement of particle

We know that $k/m = \omega^2$ where ω is said to be angular frequency.

$$\frac{d^2x}{dt^2} = \omega^2 x = 0$$

$$\frac{d^2x}{dt^2} = -\omega^2 x \qquad (1)$$

This negative sign indicates acceleration and the displacement are in opposite direction.

VELOCITY

Since, $\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$

Then, $\frac{dx}{dt} = v$

So, Acceleration = v. $\frac{dv}{dx}$

Substituting this in eq (1)

v.$\frac{dv}{dx}$ = -$\omega^2 x\, dx$

on integrating, we get , $\int v\, dv = \int -\omega^2\, x\, dx = -\omega^2 \int x\, dx$

$\frac{v^2}{2} = \frac{-\omega^2.x^2}{2}$ + c (2)

When a particle is in an extreme position, displacement will be maximum and its velocity becomes zero

X = $\pm a\, , v = 0$

0 = -$\frac{-\omega^2\, a^2}{2}$ + c

C = $\frac{-\omega^2\, a^2}{2}$ + c

Then substitute the value of c in eq(2)

Therefore, $\frac{v^2}{2} = \frac{-\omega^2\, x^2}{2} + \frac{-\omega^2\, a^2}{2}$

$V^2 = \omega^2(a^2 - x^2)$

By taking square root on both sides we get

V = $\pm\omega\, \sqrt{a^2 - x^2}$ *is the required equation of velocity of simple harmonic motion.*

AMPLITUDE

The maximum displacement of the vibrating object from the equilibrium position is called as amplitude

Y = A sinω^2

Where A denotes amplitude

ω is angular frequency

t is time period

FREQUENCY

The number of oscillations performed by an object in unit time is called as frequency. It's unit is hertz (Hz). Then it is denoted as

F = $\frac{1}{T}$

PROBLEMS

(1) A ball is released from a balloon which is 192ft above the ground and rising at 64ft/s. Find the distance over which the ball continues to rise and the time elapsed before it strikes the ground.

SOLUTION:

Let the displacement s of the ball be measured from the point at which the ball is released and s be positive upward. Also, let the time t be measured from the instant the ball is released Assume that the ball has the constant acceleration a=dv/dt = -32ft/s^2. As the velocity decreases with time, the acceleration, known as the acceleration due to gravity, is negative . The initial velocity of the ball is the same as that of the balloon and thus we have to solve the initial value problem.

a = $\frac{dv}{dt}$ = -32 (t = 0, v = 64) , (t = 0,s = 0)

Integrating this, we get v = -32t+c, which yields c = 64, and hence the force the earth exerts on the block, and vector N represents the force the plane exerts on the block. Each of the vectors W and N , magnitude W, and the weight of the block is measured in pounds.

By Newton's second law, the unbalanced force acting on the block in the x- direction equals the mass of the block times its acceleration in the x- direction. Since the unbalanced force in the x- direction has the value –kx,

m$\frac{d^2x}{dt^2}$ = -kx (1)

Where x is positive, negative or zero.

The mass m is measured in slugs and m=W/g (g = 32ft/s^2). The differential equation governing the motion of the block can be written as

$\frac{d^2x}{dt^2} + \frac{kg}{W}$x = 0 (2)

Which is a linear differential equation with steady coefficients. The A.E. $m^2 + (kg/W) = 0$ *has roots* $\pm i\sqrt{gk/W}$ and the complete solution of eq (2) is x = $c_1 \sin\sqrt{\frac{kg}{W}}\,t + c_2 \cos\sqrt{\frac{kg}{W}}\,t$ (3)

Substituting t = 0 and x=x_0, we get $c_{2=x_0}$

Differentiation of eq (3) with respect to t yields

V = $\frac{dx}{dt} = c_1\sqrt{\frac{kg}{W}}\cos\sqrt{\frac{kg}{W}}\,t - c_2\sqrt{\frac{kg}{W}}\sin\sqrt{\frac{kg}{W}}\,t$ (4)

Substituting v = 0 and t = 0, we get c_1 = 0. Hence the particular solutions of eq(2) satisfying the initial conditions we have imposed is given by

X =$x_0 \cos\sqrt{\frac{kg}{W}}t$ (5)

Equation (9) reveals that the block moves forever back and forth between the points x = x_0 and x=-x_0. *This* motion is called simple harmonic and eq(6) is the differentiation equation of the simple harmonic motion. The time required for the block to go from x = x_0 *to* $x = -x_0$ and back again is called the period of the motion. It is equation to the fundamental period of the periodic function given by

$\Psi(t) = \cos\sqrt{\frac{kg}{W}}t$

And has the value

$T=\frac{2\pi}{kg/W} = 2\pi\sqrt{\frac{W}{kg}}s$ (6)

The reciprocal of t denotes the number of complete oscillations or cycles of the block per second and is called the frequency of motion.

(2) A molecule begins from rest, a distance 10cm from a settled point O. It moves along a horizontal straight line towards O under the influence of an attractive force at 0. This force at any time varies as the distance of the particle from O. If the acceleration of the particle is 9cm/s^2 directed towards o when the particle is 1cm from O, describe the motion.

SOLUTION:

Assume that the particle starts at A at t=0. Take the fixed point o as the origin of the coordinate system and choose OA as the positive direction. Let give P a chance to be situation of the molecule whenever.. Since the magnitude of the force of attraction towards point O is proportional to the distance from the point O. We have, from Newton's law, the relation

m $\frac{d^2x}{dt^2} = -k$

$\frac{d^2x}{dt^2} - \frac{k}{m}x$ (1)

Since, the acceleration is 9cm/s^2 directed towards O when the particle is 1cm from O , we have x = 1, a = d^2x/dt^2 = -9 . Hence, from (1) k/m = 9. Thus

$\frac{d^2x}{dt^2} = -9x$ (2)

As we have a second order differential equation, two conditions are satisfied.

Since, the particle starts from rest 10cm from O , we have x = 10 , v = 0 at t = o ,Thus

$\frac{d^2x}{dt^2} = -9x$ x=10

(3)

$\frac{dx}{dt} = 0$ at t = 0

Let $\frac{dx}{dt}$ = v, so that

$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\frac{ds}{dt} = v\frac{dv}{dx}$

And eq(2) takes the form

V $\frac{dx}{dt} = -9x$

Separating the variables and integrating we get

$$v^2 = -9x^2 + c_1$$

Since v = 0, when x = 10 , c_1 = 900. Thus

$v^2 = 9(100 - x^2), v = \frac{dx}{dt} = \pm 3\sqrt{100 - x^2}$

Separating the variables and integrating we get

$\sin^{-1}\frac{x}{10} = \pm 3tc_2$,

Since, x = 10 when t = 0 and we have $c_2 = \frac{\pi}{2}$ *.Thus,*

$$\sin^{-1}\frac{x}{10} = \frac{\pi}{2} \pm 3t$$

X = 10 cos3t (4)

This shows that the particle starts at x = 10 when t = 0, then proceeds through O to the place
x = -10 , from where it returns to O again, passes through, and goes to x = 10 . The cycle then repeats over and over again. This behavior is similar to that of the bob of the pendulum swinging back and forth and is a n example of a simple harmonic motion.
Here, amplitude is 10cm, and from the graph, the period is $2\pi/3$ without the graph is to determine when the particle is at an extremity of its path, for example when x = 0 from eq(), it is soon that this will occur when
Cos 3t = 1, or 3t = 0,2π,4π,…
Hence, the first time that x = 0 is t = 0, next time t = $\frac{2\pi}{3}$, next time t = $\frac{4\pi}{3}$ and so on. The difference between the successive time is = $\frac{2\pi}{3}, which\ is\ the\ peroid\ .The\ frequency\ is\ \frac{3}{2\pi}$ cycles/s.

CONCLUSION

In this case study we have discussed about Simple harmonic motion and it's applications. Different applications problems are solved analytically. We can also calculate periodic time value of oscillating particles from origin by this method.