GENERALISED INNER DERIVATIONS

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Abstract. Let $A$ be any ring and $f$ be a generalised inner derivation of $A$. For any $x, y \in A$, we have $f(xy) = f(x)y + xh_a(y)$ for fixed element $a \in A$.

In this paper, it is shown that (i) If $f(x) = 0$ for every $r, x \in A$, $A$ is Prime ring. Then either $r = 0$ or $h_a = 0$ (ii) $f(xy) = f(xy)z + xf(yz) - f(xyz)$ $\forall x, y, z \in A$ (iii) $f(a) = f(a)b \forall b \in A$.


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Introduction

We define the generalised inner derivations of a ring. "Let $A$ be any ring. An additive mapping $f : A \rightarrow A$ is said to be generalised inner derivation if $f(xy) = f(x)y + xh_a(y)$ where $h_a : A \rightarrow A \forall x, y \in A$ & fixed element $a \in A$ $y \rightarrow h_a(y)$.

Let $G_A(a)$ be the set of all generalised inner derivations of $A$ into itself. In Thm. 2.1 we have proved that if $r(x) = 0 \forall x \in A$ then either $r = 0$ or $h_a = 0$ where $f(xy) = f(x)y + xh_a(y) \forall x, y \in A$. fixed element $a \in A$ and $f$ be the generalised inner derivation on a Prime ring $A$. In Cor. 2.3, replacing $h_a$ by $d$, we get Lemma 1. P.1093 of Posner [4]. In Thm. 2.4 we have proved that if $f \in G_A(D) A$ be any Prime ring. Then $f(xy) = f(x)y + xh_a(y) \forall x, y, z \in A$. In Cor.2.5, replacing $f$ by $d$ we get Obs.1, Remark 3 on P.90 of Bresar [1]. In Cor. 2.6 if $A$ has unity then generalised inner derivations becomes the inner derivation and vice versa. In Thm. 2.7, we have proved that if $A$ is Prime ring and $f$ is generalised inner derivation of $A$ then $f(aba) = f(a)ba \forall b \in A$, $a$ is fixed element of $A$. In Lemma 2.8, let $A$ be any Prime ring and $f \in E_G(D) A$ s.t. $xf(x) - f(x)x = 0 \forall x \in A$ Then $x(f(xa) + af(x)) = f(x)(ax + xa)$, for fixed element $a \in A$. In 3.1, replacing $h_a$ by $d$, we get Havala [2] definition of Generalised derivation. In Thm. 3.3, we have proved that if $f$ is generalised derivation of $A$ and if $a \in 0 \in A$. Then $d(ab) = ad(b) \forall b \in A$.

In the last, we have proved that if $f$ is generalised derivations of $A$, $A$ is without zero Divisors and $ab = 0$. Then $f(ba) = f(a)b + f(b)a \forall a, b \in A$.

1. Generalised Inner Derivations

In this section, we study the Generalised Inner Derivation in a Ring. Let $A$ be any Ring.

**Definition 1.1** (Generalised Inner Derivation): An additive mapping $f : A \rightarrow A$ is said to be Generalized Inner Derivation if $f(xy) = f(x)y + x[a,y]$, for fixed element $a \in A \forall x, y \in A$.

We are taking the definition as $f(xy) = f(x)y + xh_a(y)$ where $h_a : A \rightarrow A \quad y \rightarrow h_a(y) = [a,y]$ is inner derivation.

Let $G_A(a)$ = set of all generalised inner derivation of $A$ into itself.

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In this section, we take $A$ be any Prime ring.

**Definition 2.1** (Prime Ring): Let $A$ be any ring. Then $A$ is said to be Prime Ring iff

- $xy = 0 \forall a \in A$
- $x = 0 \text{ or } y = 0$

**Lemma 2.2** Let $f$ be a generalised inner derivation of a Prime ring and $r \in A$.

If $rf(x) = 0 \forall x \in A$ Then either $r = 0$ or $h_a = 0$ where $f(xy) = f(x)y + xh_a(y), \forall x, y \in A$ & fixed element $a \in A$.

**Proof.** Now $rf(x) = 0 \forall x \in A$. Replacing $x$ by $xy$, we get $rf(xy) = 0$

- $rf(x) + xh_a(y) = 0$
- $rf(x) + xh_a(y) = 0$
- $rxh_a(y) = 0$
- $either \ r = 0 \ or \ h_a = 0 (\because \ A \text{ is Prime Ring})$
- $either \ r = 0 \ or \ h_a = 0$

Hence proved.

**Corollary 2.3** Replacing $h_a$, by $d$, we have Lemma 1. P.1093 of Posner [4].

**Theorem 2.4** If $f \in G_A(D) A$ where $A$ is Prime Ring. Then $f(xy) = f(xy)z + x(f(yz) - xf(y)z \forall x, y, z \in A$.
Proof. Now \( f(xyz) - f(xy)z - xf(y)z + x(yf(z)) = (f(xy)z + xyh_{a}(z)) - f(xy)z - x(f(y)z + yh_{a}(z)) + x(f(y)z) = 0. \) Hence proved.

Corollary 2.5 Replacing \( f \) by \( d \), we get Obs.1, Remark 3, P.90 of Bresar [1].

Corollary 2.6 If \( A \) has unity. Then Generalised inner derivations become inner derivations and vice-versa. Proof. \( f(x) = f(x)y + xh_{a}(y) \) for fixed element \( a \in A \forall x, y \in A \).

Putting \( x = 1 \), we have

\[
\begin{align*}
  f(y) &= f(1)y + 1h_{a}(y) \\
  f(y) &= f(1)y + h_{a}(y) \\
  \Rightarrow & \quad f \text{ becomes inner derivations and vice versa}
\end{align*}
\]

Hence proved.

Theorem 2.7 If \( A \) is a Prime Ring and \( f \) is generalised inner derivations of \( A \). Then \( f(aba) = f(a)ba \forall b \in A \) where \( a \) be the fixed element of \( A \).

Proof. Since \( f \) is generalised inner derivation of \( A \)

\[
\Rightarrow f(x) = f(x)y + xh_{a}(y) \quad \forall x, y \in A \\& \text{fixed element } a \in A
\]

Now

\[
\begin{align*}
  f(a^{2}) &= f(a)a + ah_{a}(a) = 0 \\
  f(a^{2}) &= f(a)a
\end{align*}
\]

Replacing \( a \) by \( a + b \), we get \( f(a + b)^{2} = f(a + b)(a + b) \)

\[
\Rightarrow f(a^{2}) + f(ab + ba) + f(b^{2}) = (f(a) + f(b))(a + b)
\]

\[
\Rightarrow f(ab + ba) = f(b)a + f(a)b
\]

Let \( K \)

\[
= f(a)(ab + ba) + f(ab + ba)\]

\[
\Rightarrow K = f(a)ab + f(a)ba + (f(b)a + f(a)b)\]

\[
= f(a)ab + f(a)ba + f(b)a^{2} + f(a)ba
\]

On the other hand

\[
\begin{align*}
  K &= f(a^{2}b + 2aba + ba^{2}) \\
  K &= (f(a^{2}b + ba^{2}) + 2f(aba) \\
  &= (f(a^{2}b + f(b)a^{2}) + 2f(aba))
\end{align*}
\]

\[
\begin{align*}
  &= f(a)ab + f(b)a^{2} + 2f(aba) \quad \text{From (1) and (2) we have} \\
  2f(aba) &= 2f(a)ba \\
  \Rightarrow & \quad f(aba) = f(a)ba
\end{align*}
\]

Hence proved.

Lemma 2.8 Let \( A \) be any Prime Ring and \( f \in \text{Eng}(A) \) such that \( xf(x) - f(x)x = 0 \ \forall x \in A \). Then \( x(f(x)a + af(x)) = f(x)(ax + xa) \), for fixed element \( a \in A \).

Proof. \( xf(x) - f(x)x = 0 \) Replacing \( x \) by \( x + y \), we get

\[
\begin{align*}
  xf(y) - f(x)y = f(y)x - yf(x) & \\
  f(x)y + xh_{a}(y) = f(xy) & \quad \text{Adding both, we get}
\end{align*}
\]

Replacing \( y \) by \( xa \)

\[
\begin{align*}
  xf(xa) + xh_{a}(xa) &= f(xa)x - xaf(x) + f(x^{2}a) \\
  \Rightarrow xf(x)a + xh_{a}(x)a &= f(x)ax - xaf(x) + f(x^{2})a \\
  \Rightarrow xf(x)a + xaf(x) &= f(x)ax + f(x)xa
\end{align*}
\]
After giving the basic introduction to the generalised inner Derivation to a ring we established the results that are used to prove the most fundamental results Posner [4] Lemma 1 P.1093, Bresar [1] Obs.1, Remark 3, P.90, Havala [2] Def. P1147 as corollaries.

References