Study on Laplace Transform

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***Abstract*: An introduction to the Laplace transform is the subject of this paper. This is related to what is a Laplace transform and what it is actually, used for. The definition of the Laplace transforms and most of its important properties are mentioned with detailed evidence. This paper also contains a brief description of the inverse Laplace transform. A method used to find the time domain function. The equivalent is interpreted with a detailed description from its frequency domain. It also includes the manufacture of Laplace. Conversion of some special functions like unit step function and delta function. Some practical life applications of the Laplace transform have also been described.**

***Keywords*: Laplace Transform, Unit Step, Properties, Delta.**

1. Introduction:

This paper deals with concise inspection of what Laplace Transform is and what its application in the industry. The Laplace Transform is a specific type of integral transform.

Considering a function f(t), its corresponding Laplace Transform will be denoted as L[f(t)], where L is the Laplace Operator operated in the time domain function f(t). The Laplace Transform is also used in solving differential and integral equation. It is also used in the analysis of transient events in the electrical circuits where frequency domain analysis is used.

2. Definition of Laplace Transform:

Let f(t) be a function of a real variable t > 0. Consider the integral

$\underline{f}(p) = \int_0^\infty e^{-pt} f(t)\, dt$, where p is real.

If the integral converges for some value of p, the Laplace transform of f(t) is said to exist and given by,

$$L[f(t)] = \underline{f}(p) = \int_0^\infty e^{-pt} f(t)\, dt \quad \ldots (1)$$

Where L[f(t)] denotes the Laplace Transform.

3. Properties and Theorems of Laplace Transform:

3.1 Linearity Property:

Suppose f(t) and g(t) are two functions, a and b are the two arbitrary constants then the Laplace Transform of f(t), g(t) is $\underline{f}(p)$ *and* $\underline{g}(p)$ respectively.

$$L\,[a\, f(t) + b\, f(t)] = a\, L[f(t)] + b\, L[f(t)] \quad \ldots (2)$$

3.2 Change of Scale Property:

A linear multiplication or division of a constant with the variable is known as Scaling. Thus, if L[f(t)] $=\underline{f}(p)$ then by change of scale property,

$$L[f(at)] = \frac{1}{a}\underline{f}\left(\frac{p}{a}\right).$$

3.3 First Shifting Theorem:

Laplace Transform of first shifting theorem states that if L[f(t)] = $\underline{f}(p)$ then,

$$L\,[\,e^{-at} f(t)] = \underline{f}(p+a) \quad \ldots (3)$$

Proof: By definition,

$$L\,[\,e^{-at} f(t)] = \int_0^\infty e^{-pt}[\,e^{-at} f(t)]dt$$

$$= \int_0^\infty e^{-(p+a)t} f(t)dt$$

$$L\,[\,e^{-at} f(t)] = \underline{f}(p+a)$$

3.4 Second Shifting Theorem:

Laplace Transform of the second shifting theorem states that if L[f(t)] $=\underline{f}(p)$, then the Laplace Transform of the following function,

$$g(t) = \begin{cases} f(t-a), & when\ t > a \\ 0, & when\ t < a \end{cases}$$

is expressed as L[g(t)] = $e^{-ap}\,\underline{f}(p) \ldots (5)$

proof: By definition,

$$L[g(t)] = \int_0^\infty e^{-pt} g(t)\, dt$$

$$= \int_0^a e^{-pt} g(t)\, dt + \int_a^\infty e^{-pt} g(t)\, dt$$

$$= 0 + \int_0^\infty e^{-pt} f(t-a)\, dt$$

Now

Put t-a = u, $\therefore$ dt = du

$\therefore$ L[g(t)] = $\int_0^\infty e^{-(p+u)t}$ f(u)du

$= e^{-ap} \int_0^\infty e^{-pu} f(u)\ du$

$= e^{-ap} \int_0^\infty e^{-pt} f(t)\ dt$

$\therefore$ L[g(t)] = $e^{-ap}\underline{f}(p)$

3.5 Multiplication of powers of the variable:

The variable that has been used so far is 't'. Thus, if we multiply of t with the original function f(t), the Laplace transform can be expressed as

$$L\,[\,t^n f(t)] = (-1)^n \frac{d^n}{dp^n}\underline{f}(p) \qquad \dots (6)$$

Proof: This result can be proved by Mathematical Induction.

Step-1 To prove that the result is true for n=1.

Let L[f(t)] = $\underline{f}(p) = \int_0^\infty e^{-pt} f(t)\ dt$

Differentiating with respect to x and applying the rule of differentiation under the integral sign,

$\underline{f}'(p) = \int_0^\infty \frac{\partial}{\partial p}[e^{-pt} f(t)\,]dt$

$= -\int_0^\infty e^{-pt} t f(t)\ dt$

= - L[tf(t)]

$\therefore$ L[t f(t)] = (-1) $\frac{d}{ds}\underline{f}(p)$

Which proves the result for n=1.

Step-2 Since the result is true for n=1, it can be assumed that the result is true for n=k (natural number).

$\therefore L\,[\,t^k f(t)] = (-1)^k \frac{d^k}{dp^k}\underline{f}(p)$

Step-3 To prove this result is true for n=k+1.

From step-2,

$L\,[\,t^k f(t)] = (-1)^k \frac{d^k}{dp^k}\underline{f}(p) = \int_0^\infty e^{-pt} t^k f(t)\ dt$

Differentiating with respect to x and applying the rule of differentiation under the integral sign,

$(-1)^{k+1} \frac{d^{k+1}}{dp^{k+1}}\underline{f}(p)$

$= \int_0^\infty \frac{\partial}{\partial p}[e^{-pt} t^k f(t)\,]dt$

$= -\,L\,[\,t^{k+1} f(t)]$

$\therefore L\,[\,t^k f(t)] = (-1)^k \frac{d^k}{dp^k}\underline{f}(p).$

Which is true for n=k+1.

Therefore, by the result of Mathematical Induction, it can be said that the result is true for any value of n.

3.6 Division of variable:

If L[f(t)] =$\underline{f}(p)$, then the Laplace Transform when the function is divided by the variable can be expressed as,

$$L\,[\tfrac{1}{t} f(t)] = \int_p^\infty \underline{f}(p)dt \qquad \dots (7)$$

Proof: By definition,

$\underline{f}(p) = \int_0^\infty e^{-pt} f(t)\ dt$

Integrating on both side with respect to p between the limits p to ∞ and changing the order of the integration on the RHS,

$\int_p^\infty \underline{f}(p)dt = \int_0^\infty [\int_p^\infty e^{-pt} f(t)\ dp]dt$

$= \int_0^\infty \left[\frac{e^{-pt}}{-t} f(t)\right]_p^\infty$ dt

$= \int_0^\infty e^{-pt} \frac{f(t)}{t}\ dt$

$\therefore L\,[\tfrac{1}{t} f(t)] = \int_p^\infty \underline{f}(p)dt$

4. Laplace Transform of Derivative:

If $\underline{f}(p)$ is the transform of f(t) and if f(t) has the value f (0) when t=0, then

$$L\,[f'(t)] = p\underline{f}(p) - f(0) \quad \dots (8)$$

Proof: By definition,

$L\,[f'(t)] = \int_0^\infty e^{-pt} f'(t)dt$

Integrating by parts,

$L\,[f'(t)] = [\,e^{-pt} f(t)]_0^\infty - \int_0^\infty (-p)e^{-pt} f(t)\ dt$

$= -\,f(0) + p\int_0^\infty e^{-pt} f(t)\ dt$

$\therefore L\,[f'(t)] = p\underline{f}(p) - f(0)$

Differentiating equation (8) again with respect to variable t,

$L\,[f''(t)] = p^2 L[f(t)] - pf(0) - f'(0)$

$= p^2 \underline{f}(p) - pf(0) - f'(0)$

therefore, in general, the nth derivative can be expressed as,

$$L\,[f^n(t)] = p^n L[f(t)] - \cdots - p^2 f^{n-3}(0) - pf^{n-2}(0) - pf^{n-1}(0) \qquad \ldots (9)$$

Thus equation (9) is use to solving differential equation.

5. Laplace Transform of Integrals:

If $\underline{f}(p)$ *is the transform of* $f(t)$ *then* $L[\int_0^t f(u)\mathrm{du}] = \frac{1}{p}\underline{f}(p)$ … (10)

Proof: By definition,

$L[\int_0^t f(u)\mathrm{du}] = \int_0^\infty e^{-pt}[\int_0^t f(u)du]dt$. Integrating by parts,

$$= \left[\int_0^t f(u)\left\{\frac{e^{-pt}}{-p}\right\}\right]_0^\infty - \int_0^\infty [(\frac{e^{-pt}}{-p})\frac{d}{dt}\int_0^t f(u)\ ]$$

$$\text{But } \frac{d}{dt}\int_0^\infty f(u)\mathrm{du} = \mathrm{f(t)}$$

$$\therefore L[\int_0^t f(u)\mathrm{du}] = \int_0^\infty \frac{1}{p} e^{-pt} f(t)\, dt$$

$$= \frac{1}{p}\,\mathrm{L[f(t)]}$$

$$\therefore L[\int_0^t f(u)\mathrm{du}] = \frac{1}{p}\,\mathrm{L[f(t)]} = \frac{1}{p}\underline{f}(p)$$

The above-mentioned result can be generalized as,

$$L[\int_0^t \int_0^t \ldots\ldots\ldots\ldots \int_0^t f(u)(du)^n]$$

$$= \frac{1}{p^n}\,\mathrm{L[f(t)]} \qquad \ldots (11)$$

6. Inverse Laplace Transform:

6.1 Definition:

If L[f(t)] = $\underline{f}(p)$ then f(t) is called as the Inverse Laplace Transform of $\underline{f}(p)$ and is denoted as,

$$L^{-1}[\underline{f}(p)] = \mathrm{f(t)} \qquad \ldots (12)$$

Thus, the frequency domain function $\underline{f}(p)$ can be converted to its corresponding time domain equivalent f(t) using the Laplace Inverse operator (L^{-1}).

7. Different methods of obtaining Inverse Laplace Transform:

There are so many ways to obtain the Inverse Laplace Transform of a given frequency domain function. The choice of this method is solving a problem on Inverse Laplace Transform depends on the nature and structure of the problem itself. Often it would be noted that a single problem can be solved by different methods. A few methods have been explained below.

7.1 Using Standard Results:

A few standard results which can be used to find the inverse Laplace Transform have been tabulated below. These results can be easily proven using the standard definitions as mentioned in equation (1) and (12).

Frequency Domain Function	Inverse Laplace Transform
$\frac{1}{p}$	$L^{-1}(\frac{1}{p}) = 1$
$\frac{1}{p+a}$	$L^{-1}(\frac{1}{p+a}) = e^{-at}$
$\frac{1}{p-a}$	$L^{-1}(\frac{1}{p-a}) = e^{at}$
$\frac{1}{p^n}$	$L^{-1}(\frac{1}{p^n}) = \frac{t^{n-1}}{\Gamma n}$
$\frac{1}{p^2+a^2}$	$L^{-1}(\frac{1}{p^2+a^2}) = \frac{1}{a}\sin at$
$\frac{p}{p^2+a^2}$	$L^{-1}(\frac{p}{p^2+a^2}) = \cos at$
$\frac{1}{p^2-a^2}$	$L^{-1}\left(\frac{1}{p^2-a^2}\right) = \frac{1}{a}\sinh at$
$\frac{p}{p^2-a^2}$	$L^{-1}\left(\frac{p}{p^2-a^2}\right) = \cosh at$

Table-1

7.2 Using First Shifting Theorem:

As seen in equation (4), the First Shifting Theorem can be expressed as,

$$\mathrm{L}\,[\,e^{-at}\mathrm{f(t)}] = \underline{f}(p+a)$$

This means that if f(t) = $L^{-1}[\underline{f}(p)]$, then,

$$L^{-1}[\underline{f}(p+a)] = e^{-at}L^{-1}[\underline{f}(p)] \qquad \ldots (13)$$

7.3 Use of Partial Fraction:

It is always easier to solve a problem on Inverse Laplace Transform by expressing the given function $\underline{f}(p)$ into a sum of linear or quadratic partial fraction as,

$\underline{f}(p) = \frac{A}{(p+a)^r} + \frac{Bp+C}{(p^2+a^2)^r}$, and the use standard results given in the table-1 to find corresponding Inverse Laplace Transform.

7.4 Using Change of Scale Property:

From equation (3), the change of scale property can be denoted as,

$$L\,[\,f(at)] = \frac{1}{a}\underline{f}\left(\frac{p}{a}\right)$$

Thus, if f(t) $=L^{-1}[\underline{f}(p)]$, taking Inverse Laplace Transform,

$$L^{-1}[\tfrac{1}{a}\underline{f}(\tfrac{p}{a})] = f(at) \qquad \dots (14)$$

7.5 Convolution Theorem:

7.5.1 Definition:

If f(t) and g(t) are two functions, then the following integral,

$$\int_0^t f(u)g(t-u)\,du$$

is called the convolution of f(t) and g(t) and is denoted as f(t)*g(t).

$$\therefore f(t)*g(t) = \int_0^t f(u)g(t-u)\,du \qquad \dots (15)$$

7.5.2 Properties of Convolution:

i) f*g = g*f (Commutative Law)

ii) f*($\propto g + \beta h$) $=\propto$ (f*g) $+\beta$(f*h) (Distributive Law)

Where $\propto$ and β are arbitrary constants.

iii) (f*g)* h = f*(g*h) (Associative Law)

7.5.3 Theorem:

Let L[f(t)] = $\underline{f}(p)$ and L[g(t)] = $\underline{g}(p)$, then,

$$L^{-1}[\underline{f}(p)\underline{g}(p)] = \int_0^t f(u)g(t-u)\,du \qquad \dots (16)$$

Where f(t) = $L^{-1}[\underline{f}(p)]$ and g(t) = $L^{-1}[\underline{g}(p)]$

7.6 Using Differentiation of $\underline{f}(p)$:

If L[f(t)] $=\underline{f}(p)$, then using n=1 in equation (6),

$$L\,[t\,f(t)] = -\underline{f}'(p)$$

$\therefore$ t f(t) $= -L^{-1}[\underline{f}'(p)]$

$\therefore$ t$L^{-1}[\underline{f}(p) = -L^{-1}[\underline{f}'(p)]$

$\therefore L^{-1}[\underline{f}(p) = -\frac{1}{t}L^{-1}[\underline{f}'(p)] \qquad \dots (17)$

This method is particularly used to find the Inverse Laplace Transform of functions having tan^{-1}x, cot^{-1}x and log x terms.

7.7 Using Integration of f(t):

Equation (10) gives us the result of the Laplace Transform when the function f(t) is integrated as shown,

$$L[\int_0^t f(u)du] = \frac{1}{p}\underline{f}(p)$$

$$\therefore \int_0^t f(u)du = L^{-1}[\tfrac{1}{p}\underline{f}(p)]$$

But by definition, f(u) = $L^{-1}[\underline{f}(p)]$

$$\therefore L^{-1}[\tfrac{1}{p}\underline{f}(p)] = \int_0^t L^{-1}[\underline{f}(p)]dp \qquad \dots (18)$$

8. Laplace Transform of Periodic Functions:

Considering f(t) to be a periodic function with period a, it's Laplace Transform can be expressed as,

$$L[f(t)] = \frac{1}{1-e^{-ap}}\int_0^a e^{-pt}f(t)dt \qquad \dots (19)$$

Proof: Since f(t) is periodic with period a, f(t)= f (t + a) = f (t + 2a) = …….and so on.

$$L[f(t)] = \int_0^\infty e^{-pt}f(t)dt$$

$$= \int_0^a e^{-pt}f(t)dt + \int_a^{2a} e^{-pt}f(t)dt + \dots\dots\infty$$

Now $\int_a^{2a} e^{-pt}f(t)dt = \int_a^{2a} e^{-p(u+a)}f(u+a)du$

Where t = u + a

$$= e^{-ap}\int_0^a e^{-pu}f(u+a)du$$

$$= e^{-ap}\int_0^a e^{-pt}f(t+a)dt$$

$$= e^{-ap}\int_0^a e^{-pt}f(t)dt, \text{ since } f(t+a) = f(t)$$

Similarly, the next integral can be proved as $e^{-2ap}\int_0^a e^{-pt}f(t)dt$ and so on with all further integrals.

L[f(t)] = [1+e^{-ap} + e^{-2ap} + …….∞]$\int_0^a e^{-pt}f(t)dt$

$$\therefore L[f(t)] = \frac{1}{1-e^{-ap}}\int_0^a e^{-pt}f(t)dt$$

9. Unit Step Function:

Unit Step Function can have only two possible values either 0 or 1. Can be defined as,

U(t) = {0, $t < 0$ 1, $t \geq 0$

The function takes a jump of unit magnitude at x=0.
Taking the Laplace transform of the above function,

L [U(t)] = $\int_0^\infty e^{-pt} U(t)$dt

∴ L [U(t)] = $\int_0^\infty e^{-pt}$dt

= L (1)

∴ L [U(t)] = $\frac{1}{p}$ … (20)

9.1 Displaced Unit Step Function:
If the origin is shifted to t = a, i.e., if the function is zero before t=a, and takes a jump of unit magnitude at t=a, then the function is called the Displaced Unit Step Function.

U(t-a) = {0, $t < a$ 1, $t \geq a$

Thus, instead of taking a jump at t=0, the function now takes a jump of unit magnitude at t=a, and it is denoted as,

L [U(t-a)] = $\int_0^\infty e^{-pt} U(t-a)$dt

= $\int_0^a$ 0dt + $\int_a^\infty e^{-pt}$dt

= $\left[-\frac{e^{-pt}}{p}\right]_a^\infty$

∴L [U(t-a)] = $\frac{e^{-pt}}{p}$ … (21)

9.2 Effect of Multiplication of U(t-a):
Often in practical applications it is required to find the Laplace Transform when the time domain function itself is multiplied with the unit step function, i.e., the function will be defined as,

f(t)U(t-a) = {0, $t < a$ $f(t)$, $t \geq a$

Taking the Laplace transform on the above function,

L[f(t)U(t-a)] = $\int_0^\infty e^{-pt} f(t) U(t-a)$dt

= $\int_0^a$ 0dt + $\int_a^\infty e^{-pt} f(t)$dt

Now
let t – a = u ∴ dt = du.
When t=a, u=0. When t=∞, $u = \infty$

∴ L[f(t)U(t-a)] = $\int_0^\infty e^{-p(a+u)} f(a+u)$dt

= $e^{-ap} \int_0^a e^{-pu} f(u+a)$du

= $e^{-ap} \int_0^\infty e^{-pt} f(t+a)$dt

∴ L[f(t)U(t-a)] = e^{-ap} L [f(t+a)] … (22)

In a specific case where a=0,

∴ L[f(t)U(t-a)] = L[f(t)]

10. Delta Function (Unique Impulse Function):
It is denoted by S(t-a), where a is any constant. Delta Function is given by,

S(t-a) = {∞, $if\ t = a$ 0, $otherwise$

Thus, the integrals $\int_0^\infty S(t-a)$dt = 1
It is a periodic function,

F(t) = { $f_1(t)$, $a < t < b$ $f_2(t)$, $b < t < c$ $f_3(t)$, $t \geq c (c \leq t < \infty)$

11. Real Life Application:
i) Lance Designed by Transformation Electromagnetics and Fabricated by 3D Dielectric Printing.
ii) Exponential and Laplace Transforms on uniform time scales.
iii) Medical approach for the stream of Carbon-nano tubes suspended nano fluids is the existence of convective state using Laplace Transform.
iv) Time of death approximation from temperature readings only: A Laplace Transform method.
v) Analytical Modelling and Characterization of Electromigration Effects for Multibranch Adjacent Trees.
vi) General nonlinear model representation of high Scale Power System.
vii) Analytical Procedure per broadband high-electro Chemical Piezo electric bimorph beams with highly frequency power harvesting peter.
viii) Generalized variational principles for heat conduction models on the basis of Laplace Transform.
ix) Categorization of geological structure with help of ground penetration radar and Laplace Transform artificial neural networks.
x) Wave Propagation and transient reaction of a fluid filled FGM (Functionally Graded Material) Cylinder with rigid core using the Inverse Laplace Transform.

12. Conclusion:

Thus, this paper overall discussed what the Laplace Transform use it is. The basic use of Laplace Transformation is the change of time domain function into its frequency domain equivalent was also explained. Main properties of Laplace Transformation and a few important functions like Unit Step Function and Delta Functions we are also explained in details. It is also contained the explanation of Inverse Laplace Transformation and different methods that can be helpful to finding the Inverse Laplace Transformation. It clear that the Laplace Transformation is used in many branches of Applied Sciences.

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