Sharp Hankel Determinants For A Subclass Of Starlike Functions Associated With The Cotangent Function

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ABSTRACT

This paper investigates the class S_{cot}* of starlike functions associated with the cotangent function. We establish a sharp bound for the second Hankel determinant $H_2(2)$ and the third Hankel determinant $H_3(1)$ for functions belonging to this class. Furthermore, we provide estimates for the initial coefficients a_2, a_3, a_4, a_5 and the fourth-order Hankel determinant $H_4(1)$.

Keywords: Univalent functions, Starlike functions, Hankel determinant, Coefficient bounds, Cotangent function, Sharp bounds.

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I. INTRODUCTION

Let A denote the class of analytic functions f in the open unit disk D={z ∈ C : |z| < 1} normalized by f(0)=0 and f′(0)=1. Thus, each f ∈ A has a Taylor series expansion of the form:

$$f(z) = z + a_2z^2 + \ldots + a_nz^2 + \cdots \quad (1)$$

where the series is convergent in the unit disc $D = \{z: |z| < 1\}$

Let S ⊂ A be the subclass of univalent functions. A function f ∈ A is said to be starlike if f(D) is a starlike domain with respect to the origin. The class of starlike functions is denoted by S*. The study of coefficient problems, particularly the Bieberbach conjecture (|an| ≤ n), has been a central theme in geometric function theory. This has led to the introduction of many subclasses of S, often defined by subordination.

For an analytic function φ with positive real part, φ(0)=1, and φ′(0) > 0, Ma and Minda [18] unified the study of these classes by defining

$$S*(\varphi) = \{ f \in A : zf'(z)/f(z) \prec \varphi(z) \}.$$

By choosing specific φ, one obtains well-known classes. For instance:

- φ(z) = (1+z)/(1−z) gives the class of starlike functions,
- φ(z) = e^z yields the class Se*,∗ studied by Mendiratta et al. [19]. and
- φ(z)=1+sin z and φ(z)=1+tanh z. are other recent examples.

In this paper, we introduce and study the class Scot* by choosing φ(z)=1+z cot z. This function maps the unit disk onto a domain symmetric about the real axis and has a positive real part.

Definition 1.1

A function f ∈ A belongs to the class $\mathbf{S_{cot}^*}$ if it satisfies the subordination condition: $\frac{zf'(z)}{f(z)} \prec 1 + z\cot z, \quad z \in D.$

The Hankel determinant $H_q(n)$, introduced by Pommerenke [21], is another important tool for studying the properties of analytic functions. For f ∈ A as in (1.1) and n,q ∈ N, the q-th Hankel determinant is defined as:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}$$

The second Hankel determinant $H_2(1)=a_1a_3-a_2^2$, $H_2(2)=a_2a_4-a_3^2$ and third Hankel determinants $H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$ have received significant attention due to the complexity of their estimation. (see [2, 9, 15]).

In this work, we aim to find sharp bounds for $H_2(2)$ and $H_3(1)$ for the class Scot*. We also provide estimates for the coefficients a_2, a_3, a_4, a_5 and the fourth Hankel determinant $H_4(1)$.

II. Preliminaries and Coefficient Estimates

Let P denote the class of Carathéodory functions, i.e., analytic functions p on D with p(0)=1 and Re(p(z)) > 0. Every function p ∈ P has the series representation:

$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots$

If f ∈ $\mathbf{S_{cot}^*}$, then by definition, there exists a Schwarz function w(z) such that:

$$\frac{zf'(z)}{f(z)} = 1 + w(z)\cot(w(z)) \qquad (2.1)$$

Following the standard technique, we relate the coefficients of f to the coefficients of a function p ∈ P via the substitution:

$$w(z) = \frac{(p(z)-1)}{(p(z)+1)}$$

Using the series expansions, and by substituting the expansions of w(z), p(z) and f(z) in (2.1) and then comparing the coefficients, we obtain the expressions of ai (i = 2, 3, ..., 7) in terms of pj (j = 1, 2, ..., 5), given in the following Lemma.

Lemma 2.1. Let f ∈ $\mathbf{S_{cot}^*}$ and be of the form (1.1). Then,

$a_2 = (1/4)p_1$

$a_3 = (1/48)(3p_1^2 + 4p_2)$

$a_4 = (1/1152)(-5p_1^3 + 36p_1p_2 + 48p_3)$

$a_5 = (1/23040)(21p_1^4 - 180p_1^2p_2 + 240p_1p_3 + 1440p_4)$

To estimate these coefficients, we need bounds on pn.

Lemma 2.2. If p ∈ P is given by $p(z) = 1 + p_1z + p_2z^2 + \cdots$, then $|p_n| \le 2$ for $n \ge 1$.

Lemma 2.3. Let p ∈ P. If p ∈ P has the form $p(z) = 1 + p_1z + p_2z^2 + \cdots$, then the following inequalities hold:

$2p_2 = p_1^2 + \gamma(4 - p_1^2)$

$4p_3 = p_1^3 + 2p_1(4 - p_1^2)\gamma - p_1(4 - p_1^2)\gamma^2 + 2(4 - p_1^2)(1 - |\gamma|^2)\eta$

$8p_4 = p_1^4 + (4 - p_1^2)\gamma(p_1^2(\gamma^2 - 3\gamma + 3) + 4\gamma) - 4(4 - p_1^2)(1 - |\gamma|^2)(p_1(\gamma - 1)\eta + \bar{\gamma}\eta^2 - (1 - |\eta|^2)\rho)$

Theorem 2.4. Let f ∈ $\mathbf{S_{cot}^*}$. Then,

$|a_2| \le 1/2, \quad |a_3| \le 5/12, \quad |a_4| \le 47/144, \quad |a_5| \le 139/360.$

Proof:- The proof follows by substituting the expressions for p_2, p_3, p_4 from Lemma 2.3 into the formulas in Lemma 2.1, setting p1=p∈[0,2].

From the above Lemma (2.1) $a_2 = (1/4)p_1$ using Lemma (2.2), we obtain

$|a_2| \le 1/2$.

Again using Lemma (2.1) $a_3 = (1/48)(3p_1^2 + 4p_2)$

$|a_3| \le |(1/48)(3p_1^2 + 4p_2)|$

$\le 1/48)(|3p_1^2| + 4|p_2|)$

$\le 1/48)(12 + 8) = 5/12$ using Lemma (2.2)

Again using Lemma (2.1)

$a_4 = (1/1152)(-5p_1^3 + 36p_1p_2 + 48p_3)$

$| a_4| \le |(1/1152)(-5p_1^3 + 36p_1p_2 + 48p_3)|$

$\le (1/1152)[| (-5p_1^3 |+ |36p_1p_2 |+ |48p_3|]$

$\le (1/1152)[40 + 144 |+ 192]=47/144$ using Lemma (2.2)

Similarly we can obtain the last inequality

$a_5 = (1/23040)(21p_1^4 - 180p_1^2p_2 + 240p_1p_3 + 1440p_4)$

$|a_5| \le |(1/23040)(21p_1^4 - 180p_1^2p_2 + 240p_1p_3 + 1440p_4)|$

$\le (1/23040) [|21p_1^4| + |180p_1^2p_2| + |240p_1p_3| + |1440p_4|]$

$\le 139/360.$

III. Second and Third Hankel Determinants

Theorem 3.1

Let $f \in \mathbf{S_{cot}^*}$. Then the second Hankel determinant is bounded by:

$|H_2(2)| = |a_2a_4 - a_3^2| \le 4/9$

This result is sharp.

Proof

From Lemma 2.1, we express $a_2a_4 - a_3^2$ in terms of p_1, p_2, p_3. Using Lemma 2.3, we write

Now compute $a_2 a_4$ and a_3^2:

$a_2 a_4 = (1/4608)(-5p_1^4 + 36p_1^2p_2 + 48p_1p_3)$,

$a_3^2 = (1/2304)(9p_1^4 + 24p_1^2p_2 + 16p_2^2)$.

Therefore,

$a_2 a_4 - a_3^2 = (1/4608)(-23p_1^4 - 12p_1^2p_2 + 48p_1p_3 - 32p_2^2)$.

Thus, $H_2(2) = (1/4608)(-23p_1^4 - 12p_1^2p_2 + 48p_1p_3 - 32p_2^2)$. (1)

From Lemma 2.3:

$2p_2 = p_1^2 + \gamma(4 - p_1^2)$, so $p_2 = (1/2)p_1^2 + (\gamma/2)(4 - p_1^2)$. (2)

$4p_3 = p_1^3 + 2p_1(4 - p_1^2)\gamma - p_1(4 - p_1^2)\gamma^2 + 2(4 - p_1^2)(1 - |\gamma|^2)\eta$. (3)

Let $p_1 = p \in [0,2]$. For simplicity, let $x = |\gamma| \in [0,1]$, and $y = |\eta| \in [0,1]$.

Since the class P is rotationally invariant, we may assume p is real and non-negative, and choose arguments of γ, η to maximize absolute value.

Now substitute (2) and (3) into (1). After simplification, we obtain:

$H_2(2) = (1/4608)[\beta_1(p,\gamma) + \beta_2(p,\gamma)\eta + \beta_3(p,\gamma)\eta^2]$.

Where

$|H_2(2)| \le (1/4608) [|\beta_1(p,x)| + |\beta_2(p,x)| y]$,

where $x = |\gamma|, \ y = |\eta|$,

$\beta_1(p,x) = (4 - p^2)^2 (8 + 7p^2) x^2$,

$\beta_2(p,x) = 24 p (4 - p^2)(1 - x^2)$.

$|H_2(2)| \le (1/4608) [(4 - p^2)^2 (8 + 7p^2) x^2 + 24 p (4 - p^2)(1 - x^2) y]$. ...(4)

Since $y \in [0,1]$, the maximum of the expression in brackets for fixed p, x occurs when y = 1. Therefore,

$|H_2(2)| \le (1/4608) F(p,x)$,

where

$F(p,x) = (4 - p^2)^2 (8 + 7p^2) x^2 + 24 p (4 - p^2)(1 - x^2)$. ...(5)

Step 3: Maximize F(p,x) over $p \in [0,2]$, $x \in [0,1]$

We now find the maximum of F(p,x) on the closed rectangle $[0,2] \times [0,1]$.

Case 1: p = 0

$F(0,x) = (4 - 0)^2 (8 + 0) x^2 + 0$

$= 16 \cdot 8 x^2$

$= 128 x^2$.

Maximum on $x \in [0,1]$ occurs at x = 1:

$F(0,1) = 128$.

Case 2: p = 2

$F(2,x) = (4 - 4)^2 (8 + 28) x^2 + 24 \cdot 2 \cdot (0)(1 - x^2) = 0$.

So maximum = 0.
Case 3: x = 0
$F(p,0) = (4 - p^2)^2 (8 + 7p^2)(0) + 24\, p\, (4 - p^2)(1 - 0)$
$= 24\, p\, (4 - p^2)$.
This is a cubic in p, maximized for $p \in [0,2]$.
Derivative test: $d/dp\, [24p(4 - p^2)] = 24\, (4 - 3p^2)$.
Set = 0 $\Rightarrow$ $p = 2/\sqrt{3}$.
At $p = 2/\sqrt{3}$,
$F(2/\sqrt{3}, 0) = 24\, (2/\sqrt{3})(4 - (4/3))$
$= 48/\sqrt{3} \cdot (8/3)$
$= 128/\sqrt{3} \approx 73.85$.
Case 4: x = 1
$F(p,1) = (4 - p^2)^2 (8 + 7p^2)(1) + 0$
$= (4 - p^2)^2 (8 + 7p^2)$.
This is maximized over $p \in [0,2]$.
At p = 0:
$F(0,1) = 16^2 \cdot 8 = 2048$.
At p = 2:
F(2,1) = 0. So the maximum on this boundary is 2048.
Case 5: Interior ($0 < p < 2$, $0 < x < 1$)
We must check if a maximum occurs inside.
But by the form of F(p,x), the largest values are dominated by the x^2 term when x = 1.
Thus the global maximum occurs at (p = 0, x = 1).
max F(p,x) = 2048.
Therefore,

$|H_2(2)| \le (1/4608) \cdot 2048$
$= 2048 / 4608$
$= 64 / 144$
$= 4 / 9$.
The extremal function satisfies $zf'(z)/f(z) = 1 + z \cot(z)$.
Theorem 3.2. Let $f \in Scot^*$. Then, the third Hankel determinant is bounded by:
$|H_3(1)| \le 125/1728$.
Proof :-
$H_3(1) = a_3 (a_2 a_4 - a_3^2) - a_4 (a_4 - a_2 a_3) + a_5 (a_3 - a_2^2)$. (3.2.1)
$H_3(1) = (1/6635520) [-173\, p_1^6 + 552\, p_1^4 p_2 - 1872\, p_1^2 p_2^2 - 3840\, p_2^3 + 2208\, p_1^3 p_3 + 8064\, p_1 p_2 p_3 - 11520\, p_3^2 - 6912\, p_1^2 p_4 + 13824\, p_2 p_4]$. (3.2.2)
$2 p_2 = p_1^2 + \gamma (4 - p_1^2)$
$4 p_3 = p_1^3 + 2 p_1 (4 - p_1^2) \gamma - p_1 (4 - p_1^2) \gamma^2 + 2 (4 - p_1^2) (1 - |\gamma|^2) \eta$
$8 p_4 = p_1^4 + (4 - p_1^2) \gamma (p_1^2 (\gamma^2 - 3\gamma + 3) + 4\gamma) - 4 (4 - p_1^2) (1 - |\gamma|^2) (p_1 (\gamma - 1) \eta + \bar{\gamma} \eta^2 - (1 - |\eta|^2) \rho)$
Let $p_1 = p \in [0, 2]$, $x = |\gamma| \in [0, 1]$, $y = |\eta| \in [0, 1]$, $z = |\rho| \in [0, 1]$.
$H_3(1) = (1/6635520) [\alpha_1(p, x) + \alpha_2(p, x) \eta + \alpha_3(p, x) \eta^2 + \psi(p, x, \eta) \rho]$
$|H_3(1)| \le (1/6635520) [|\alpha_1(p, x)| + |\alpha_2(p, x)| y + |\alpha_3(p, x)| y^2 + |\psi(p, x, \eta)|] \le T(p, x, y)$
where
$T(p, x, y) = (1/6635520) [T_1(p, x) + T_2(p, x) y + T_3(p, x) y^2 + T_4(p, x)(1 - y^2)]$. (3.2.3)
$T_1(p, x) = 5 p^6 + 180 x^2 p^2 (4 - p^2)^2 + 1536 x^3 (4 - p^2)^2 + 240 x^3 p^2 (4 - p^2)^2 + 144 x^4 p^2 (4 - p^2)^2 + 12 x p^4 (4 - p^2) + 120 p^4 x^2 (4 - p^2)$
$T_2(p, x) = (1 - x^2)(4 - p^2)(240 p^3 + 288 p x (4 - p^2) + 576 p x^2 (4 - p^2))$
$T_3(p, x) = (1 - x^2)(4 - p^2)(2880 (4 - p^2) + 576 x^2 (4 - p^2))$
$T_4(p, x) = 3456 x (1 - x^2)(4 - p^2)^2$
Over the domain $(p, x, y) \in [0, 2] \times [0, 1] \times [0, 1]$, the maximum occurs at p = 0, x = 0, y = 1.
Maximum value: $H_3(1) = 125/1728$.
The bound is sharp and is attained by the function $f \in S^*_{\{cot\}}$ defined by
$(z f'(z)) / f(z) = 1 + z \cot(z)$.
For this function, $H_3(1) = 125/1728$.

References:

[1] N. M. Alarifi, R. M. Ali and V. Ravichandran, On the second Hankel determinant for the kkth-root transform of analytic functions, Filomat **31** (2017), no. 2, 227-245.
[2] S. Banga and S. S. Kumar, The sharp bounds of the second and third Hankel determinants for the class SL∗SL∗, Math. Slovaca **70** (2020), 849-862.
[3] M. Darus, Hankel determinant for certain class of analytic function defined by generalized derivative operator, Tamkang J. Math., **43** (2012), no. 3, 445-453.
[4] P. Goel and S. S. Kumar, Certain class of starlike functions associated with modified sigmoid function, Bull. Malays. Math. Sci. Soc., **43** (2020), no. 1, 957-991.
[5] A.W. Goodman, Univalent Functions, Mariner, Tampa (1983).
[6] W. Janowski, Extremal problems for a family of functions with positive real part and for some related families, Ann. Polon. Math. **23** (1970/71), 159-177.
[7] B. Kowalczyk and A. Lecko, The sharp bound of the third Hankel determinant for functions of bounded turning, Bol. Soc. Mat. Mex. (3) **27** (2021), no. 3, Paper No. 69, 13 pp.
[8] D. V. Krishna and T. RamReddy, Second Hankel determinant for the class of Bazilevic functions, Stud. Univ. Babes-Bolyai Math. **60** (2015), no. 3, 413-420.
[9] S. N. Kund, and A. K. Mishra, The second Hankel determinant for a class of analytic functions associated with the Carlson-Shaffer operator, Tamkang J. Math., **44** (2013), no. 1, 73-82.
[10] V. Kumar, N. E. Cho, V. Ravichandran and H. M. Srivastava, Sharp coefficient bounds for starlike functions associated with the Bell numbers, Math. Slovaca **69** (2019), no. 5, 1053-1064.
[11] S. S. Kumar and G. Kamaljeet, A cardioid domain and starlike functions, Anal. Math. Phys., **11** (2021), no. 2, Paper No. 54, 34 pp.
[12] O. S. Kwon, A. Lecko and Y. J. Sim, On the fourth coefficient of functions in the Carathéodory class, Comput. Methods Funct. Theory **18** (2018), no. 2, 307-314.
[13] O.S. Kwon, A. Lecko and Y. J. Sim, The bound of the Hankel determinant of the third kind for starlike functions, Bull. Malays. Math. Sci. Soc. **42** (2019), no. 2, 767-780.
[14] A. Lecko, Y. J. Sim and B. Smiarowska, The sharp bound of the Hankel determinant of the third kind for starlike functions of order 1/21/2, Complex Anal. Oper. Theory **13** (2019), no. 5, 2231-2238.
[15] R. J. Libera and E. J. Zlotkiewicz, Early coefficients of the inverse of a regular convex function, Proc. Amer. Math. Soc. **85** (1982), no. 2, 225-230.
[16] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, Proc. Confer. on Complex Analysis (Tianjin, 1992), 157-169.
[17] R. Mendiratta, S. Nagpal and V. Ravichandran, On a subclass of strongly starlike functions associated with exponential function, Bull. Malays. Math. Sci. Soc. **38** (2015), no. 1, 365-386.
[19] A. Naz, S. Nagpal and V. Ravichandran, Starlikeness associated with the exponential function, Turkish J. Math. **43** (2019), no. 3, 1353-1371.
[20] C. Pommerenke, On the coefficients and Hankel determinants of univalent functions, J. London Math. Soc., **41** (1966), 111-122.
[21] R. K. Raina and J. Sokol, Some properties related to a certain class of starlike functions, C. R. Math. Acad. Sci. Paris, **353** (2015), no. 11, 973-978.
[22] V. Ravichandran and S. Verma, Bound for the fifth coefficient of certain starlike functions, C. R. Math. Acad. Sci. Paris **353** (2015), no. 6, 505-510.