Class of Accelerated Sequential Procedure For Fixed-Size Confidence Region

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Abstract: In this paper, a class of accelerated sequential procedures is constructed for fixed size confidence region of parameter in the presence of unknown nuisance parameter under the set-up of distributional relationship developed by Chaturvedi, A., Pandey S.K., Gupta , (1991)[2], and results of Hall(1983) [3] was proven.

Index word: Accelerated, Chi-Square distribution, Confidence-Region

1. INTRODUCTION

In order to construct fixed-range confidence interval for a normal mean, assuming the variance to be unknown, Hall [3](1983) proposed an accelerated’ sequential Procedure which combines the rates of two-stage and purely sequential procedures and also is more flexible in nature because the number of sampling stages can be reduced only by introducing finite number of observations. Several other experimenters have also developed and studied the same for other distributions also.

In the present Chapter, we develop the classes of ‘accelerated’ sequential procedures to construct fixed size confidence region for the parameter \( \theta \) for the bounded risk point estimation. The set up [2] of the problem is:

\[
X_1, ..., X_n \text{ be a random sample of size } n \geq t + 1, \text{from a t variate continuous population, with parameter } \theta \text{ of order } t \times 1 \text{ of interest and } \Psi \text{ a scalar unknown parameter, let } (\theta', \Psi') \in R^t \times R^r. \text{ The estimators of } \theta \text{ and } \Psi \text{ are } \hat{\theta}_n = \hat{\theta}(X_1, ..., X_n) \text{ and } \hat{\Psi}_n = \hat{\Psi}(X_1, ..., X_n). \text{ The following hypotheticals are made}
\]

(A1): A known positive definite matrix \( Q \), of order \( r \) by \( t \), a number \( \delta \in (0,1) \) and a positive integer \( r \geq 1 \) exist, then for all \( n > 0 \),

\[
t_{n}[\psi^{-1}(\theta_n - \theta)']Q(\theta_n - \theta)]\delta \sim \chi^2(\delta)
\]

(A2): \( \hat{\theta}_n \) and \( \hat{\Psi}_n \) are independent for all values of \( n \).

(A3): For integers \( s \geq 1 \) then for all \( n \) greater than or equal to \( s + 1 \),

\[
r(n - s)\hat{\Psi}_n/\Psi = \sum_{j=s+1}^{n}Z_j^{(r)}
\]

where \( Z_j^{(r)} \)'s are iid rv’s with \( Z_j^{(r)} \sim \chi^2(\delta) \).

(A4): \( \hat{\Psi}_n \) is a consistent estimator of \( \psi \).

For specified \( d(>0) \) and \( \delta \in (0,1) \) to construct a confidence region \( R_n \) (which may be interval, ellipsoidal or spherical) for \( \theta \), of maximum width \( 2d \) and \( P(\hat{\theta} \in R_n) \geq \alpha \). We define

\[
R_n = \left[ z: \left((\hat{\theta}_n - z) \cdot Q(\hat{\theta}_n - z)\right)^{\delta} \leq a^2 \right] \quad \cdots \cdots \cdots \cdots \cdots (1.1)
\]

when \( t \geq 2 \), for \( \delta = 1, R_n \) is an ellipsoidal confidence region and for \( \delta = 1, Q = I_{txr}, R_n \) reduces to a spherical region. Moreover, for \( t = \delta = 1 \) and \( l = I_{1 \times 1} = 1 \), since

\[
P(\theta \in R_n) \equiv P\left(\left| \hat{\theta}_n - \theta \right| < c^2 \right) = P\left(\left(\hat{\theta}_n - \theta\right)^2 \leq c^2 \right)
\]
the results based on the region (1.1) are same as those based on a confidence interval of width 2d.

Denoting by \( G^{(r)}(\cdot) \), cdf of a \( X^{(r)}(\text{Chi square}) \) random variable \( \cdot \), and utilizing \( (A_1) \), we obtain from (1.1)

\[
P(\theta \in R_n) = P\left[\left((\hat{\theta}_n - \theta)Q(\hat{\theta}_n - \theta)\right)^{\delta} \leq d^2\right]
= p[X^{(r)}_n \leq n\psi^{-1}d^2]
= G^{(r)}(n\psi^{-1}d^2)
= \cdots \cdots \cdots \cdots \cdots \cdots \cdots (1.2)
\]

Let 'a' be the constant satisfying the relation

\[G^{(r)}(a^{-2}) = \alpha \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots (1.3)\]

Using monotonicity property of distribution function, it follows from (1.2) and (1.3) that, for known \( \psi \); in order to achieve \( P(\theta \in R_n) \geq \alpha \), the (fixed) sample size required is the smallest positive integer \( n \geq n^* \), where

\[n^* = \left(\frac{a}{d}^2\right)\psi\]

But, in the ignorance of \( \psi \), no fixed sample size procedure achieves the goals of pre-assigned width and coverage probability simultaneously for all values of \( \psi \). To meet the requirements, we adopt the following class \( \mathcal{C}_{\lambda}^{**} \) of “accelerated” sequential procedures.

Start with a sample of size \( m \geq \max \cdot (t + 1, s + 1) \), so as to satisfy \( m = o(d^{-2}) \) as \( d \to 0 \) and \( \lim \sup \, (m/n^*) \) and fix two constants \( L(>0) \) and \( \eta \in (0,1) \). Start taking observations sequentially with the stopping time \( N_t \) defined by

\[N_t = \text{Inf} \left\{ n_t \geq m, n_t \geq \eta \left(\frac{a}{d}^2\right)^2 \hat{\psi}_n \right\}. \quad (1.4)\]

Based on these \( N_t \) observations compute \( \hat{\psi}_n \). Then jump ahead and collect \( N_2 \) observations, where;

\[N_2 = \left[\left(\frac{a}{d}^2\right)^2 \hat{\psi}_n + L\right]^+ + 1 \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots (1.5)\]

Let \( N = \max(\, N_1, N_2) \) and construct the region for \( \theta \).

\[\bar{R}_n = \left\{ z: \left((\hat{\theta}_n - z)^{\delta} \cdot Q(\hat{\theta}_n - z)\right)^{\delta} \leq d^2 \right\} \]

we first establish some basic lemmas

**Lemma 1.1:** \( \lim_{d \to 0} N_1 = \lim_{d \to 0} N_2 = \infty \)

Proof: The proof is an immediate consequence of the definition (1.5) and (1.6) of \( N_1 \) and \( N_2 \).

**Lemma 1.2:** \( \lim_{d \to 0} \left( \frac{N}{n^*} \right) = 1 \) a.s.

Proof: From (1.5), we notice that:

\[
\eta \left(\frac{a}{d}^2\right)^2 \hat{\psi}_{N_1} \leq N_1 \leq \eta \left(\frac{a}{d}^2\right)^2 \hat{\psi}_{N_1-1}^* + 1, \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots (1.7)
\]

or

\[
\left(\hat{\psi}_{N_1}/\psi\right) \leq \left( N_1 \eta n^* \right) \leq \left( \hat{\psi}_{N_1} - 1^{\psi} \right) + (\eta n^*)^{-1}, \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots (1.8)
\]

using the fact that \( \hat{\psi}_n \) is a consistent estimator of \( \psi \) and applying Lemma 1.1, we obtain from (1.8)

\[
\lim_{d \to 0} \left( \frac{N_1}{\eta n^*} \right) = 1 \quad \text{a.s.}
\]

Similarly using (1.6), it can be shown that

\[
\lim_{d \to 0} \left( \frac{N_2}{n^*} \right) = 1 \quad \text{a.s.}
\]

The lemma now follows from the definition of \( N \).
Lemma 1.3: As \( d \to 0, (\eta n^*)^{-\frac{1}{2}}(N_1 - \eta n^*) \overset{L}{\to} N(0, 2q^{-1}) \)

Proof: Using (A3), we can rewrite the stopping rule (1.5) as follows

\[ N_1 = \text{Inf} \cdot \left[ n_1 \geq m \sum_{j=1}^{n_1-s} q^{-1}z_j^{(q)} \leq (n_1 - s) n_1/\eta n^* \right]. \tag{1.9} \]

Let us define a new stopping variable \( N_1^* \) by

\[ N_1^* = \text{Inf} \cdot \left[ n_1 \geq m - s: \sum_{j=1}^{n_1-s} \left( q^{-1}z_j^{(q)} \right) \leq n_1^2 \cdot (1 + n_1^{-1})/\eta n^* \right] \tag{1.10}. \]

It follows from Lemma 1 of Swanepol and Vanwyk [4] (1982) that the stopping variables \( N_1 \) and \( N_1^* \) follows the same probability distribution. Comparing (1.10) with equation (1.1) of Woodrooffe [5] (1977), we obtain in his notations \( \alpha = 2, \beta = 1, \mu = 1 \) and \( \tau^2 = 2q \). The lemma now follows from a result of Bhattacharya and Mallik [1] (1973) that

\[ (\eta n^*)^{-\frac{1}{2}}(N_1 - \eta n^*) \overset{L}{\to} N(0, \beta^2 \tau^2 \mu^{-2}) \]

Lemma 1.4: For all \( m \geq \{t, s + 2q-1\} \), as \( d \) tends to 0

\[ E(N_1) = \eta n^* + v - (s + 2q^{-1}) + o(1), \]

where \( v \) is specified.

Proof: In the notations of Woodrooffe [5] (1977) \( a = q/2, \lambda = \eta n^*, L(n) = 1 + 3n^{-1} \) and \( L_0 = S \). The lemma now follows from his Theorem 2.4 that, for all \( m > s + 2q^{-1} \) as \( d \to 0 \)

Lemma 1.5: For all \( m > \max\{t, s + 2q^{-1}\} \) as \( d \to 0 \).

\[ E(N) = n^* + L - \eta^{-1}(s + 2q^{-1}) + o(1), \quad \text{(1.11)} \]

\[ \text{Var} \cdot (N) = \left( 2n^*/\eta q \right) + o(d^{-2}), \quad \text{(1.12)} \]

and, for specified \( \gamma(>0) \).

\[ E[N - E(N)]^\gamma = O(d^{-\gamma}) \ldots \ldots \ldots \quad \text{(1.13)} \]

Proof: The stopping rule (1.5) can be re-written

\[ N_1 = \text{Inf} \cdot \left[ n_1 \geq m: q(n_1 - s) \frac{\hat{\psi}_1}{\psi} \leq \frac{d^2 qN_1(n_1 - s)}{\eta a^2 \psi} \right]. \]

Let us consider the difference.

\[ D_d = \frac{d^2 qN_1(N_1 - s)}{\eta a^2 \psi} - q_1(N_1 - s) \frac{\hat{\psi}_1}{\psi}. \tag{1.14} \]

It follows from Woodrooffe [5] (1977) that the mean of the asymptotic distribution of \( D_d \) is \( v \). Let us define

\[ D_d^* = \eta [q(N_1 - s)]^{-1}(a/d)^2 \psi D_d \]

Since

\[ q(N_1 - s) \frac{\hat{\psi}_{N_1}}{\psi} = \sum_{j=1}^{N_1-s} z_j^{(q)} \quad \text{and} \]

\[ q(N_1 - s - 1) \frac{\hat{\psi}_{N_1-1}}{\psi} = \sum_{j=1}^{N_1-s-1} z_j^{(q)}, \]

one concludes that;
\[ q(N_1 - s) \frac{\hat{\psi}_{N_1}}{\psi} \to q(N_1 - s - 1) \frac{\hat{\psi}_{N_1-1}}{\psi}. \]

Hence, from (1.7), (1.14) and (1.15),
\[ D_d^* = \eta \{ q(N_1 - s) \}^{-1} (a/d)^2 \psi \left[ \frac{d^2 q(N_1 - s)}{\eta^2 \psi} - q(N_1 - s) \frac{\psi_{N_1}}{\psi} \right] \leq \eta \{ q(N_1 - s) \}^{-1} (a/d)^2 \psi \left[ \frac{d^2 q(N_1 - s)}{\eta \psi} - q(N_1 - s - 1) \frac{\psi_{N_1}}{\psi} \right] \]
\[ \leq \eta \{ q(N_1 - s) \}^{-1} (a/d)^2 \psi \left[ \frac{d^2 q(N_1 - s)}{\eta a^2 \psi} - q(N_1 - s - 1) \frac{\psi_{N_1}}{\psi} \right] = N_1 - (N_1 - S - 1) = s + 1 \]

Furthermore, assuming (1.7) we get.
\[ \eta \{ q(N_1 - s) \}^{-1} (a/d)^2 \psi \left[ \frac{d^2 q(N_1 - s)}{\eta a^2 \psi} - q(N_1 - s) \frac{\psi_{N_1}}{\psi} \right]. \]
\[ \geq \eta \{ q(N_1 - s) \}^{-1} (a/d)^2 \psi \left[ \frac{d^2 q(N_1 - s)}{\eta a^2 \psi} - q(N_1 - s) \frac{\psi_{N_1}}{\psi} \right]. \]
\[ = 0 \]

Thus, \( 0 \leq D_d^* \leq s + 1 \) and hence from dominated convergence theorem \( E(D_d^*) \to \nu \) as \( d \to 0 \). Utilizing this result and Lemma 4.9, we obtain for all
\[ m > \max \{ t, s + 2q^{-1} \}, \text{ as } d \to 0. \]
\[ E(D_d^*) = \nu = E[N_1 - \eta(a/d)^2 \hat{\psi}_{N_1}] \]
\[ E[(a/d)^2 \hat{\psi}_{N_1}] = \eta^{-1} E(N_1 - \nu). \]
\[ = n^* - \eta^{-1} (s + 2q^{-1}) + o(1). \]

and (1.11) follows from the definition of \( N \).

By the definition of \( N \),
\[ \text{Var} \{ N \} = \eta^{-2} \text{Var} \cdot (N_t). \]
Let \( \eta(N_t) = (\eta n^*)^{-1/2} (N_t - \eta n^*) \).

It follows from theorem 2.3 of Woodroofe [5] (1977) that \( h^2(N_t) \) is uniformly integrable for all \( m > s + 2q^{-1} \).

Now utilizing Lemma 4.8, we obtain for all \( m > s + 2q^{-1} \), as \( d \to 0 \)
\[ \text{Var} \{ N \} = \eta^{-2} [2q^{-1} \eta n^* \{ 1 + o(1) \}]. \]
and (1.12) holds.
The proof of (1.13) is similar to that of result (3) of Hall (1983).

**Result:** The main result is now stated and proved in the following theorem:
Theorem: For all \( m > (s + 2q^{-1}), \text{and sufficiently small } d, \text{ say } d \leq d_0 \)
\[ P(\theta \in R_N) \geq \alpha; \quad \text{if } L > \eta^{-1} \{ s - (2q)^{-1} \cdot (r - a^2 - 6) \} \]

Proof: Utilizing \( (A_2) \), the coverage probability associated with the sampling scheme (1.5) – (1.6) is
\[ P(\theta \in R_N) = \sum_{n=1}^{\infty} P\left[ n\psi^{-1}\{(\hat{\theta}_n - \theta)\} \cdot Q(\hat{\theta}_n - \theta) \right] \]

\[ \leq a^2(n/n^*); N = n \]

\[ = \sum_{n=1}^{\infty} G^{(r)}(a^2 n/n^*) \cdot P(N = n) \]

\[ = E[G^{(r)}(a^2 N/n^*)]. \]

Expanding \( G^{(r)}(.) \) around \( a^2 \) by second-order Taylor's series, we obtain for

\[ |a^2 - W| \leq a^2[(N/n^*) - 1] \]

\[ P(\theta \in R_N) = G^{(r)}(a^2) + a^2 G^{(r)'}(a^2) E[(N/n^*)] + (a^4/2) E[(N/n^*) - 1]^2 G^{(r)''}(a^2) + \xi d^r \]

where the remainder term \( \xi d = O(a^6E|N - E(N)|^3) \).

Denoting by \( g^{(r)}(.) \), the p.d.f. of a \( x^2 \) r.v, we note that

\[ G^{(r)'}(x) = g^{(r)}(x) \]

and

\[ G^{(r)''}(x) = \left( \frac{r}{2} - 1 \right) x^{-1} - \frac{1}{2} \] \( g^{(r)}(x). \)

Hence applying Lemma 1.5., we obtain for all \( m > s + 2q^{-1} \).

\[ P \left( \frac{\theta \in R_N}{\theta} \right) = \alpha + \left( \frac{a^2}{n^*} \right) \{ L - \eta^{-1}(s + 2q^{-1}) + o(1) \} \]

\[ \cdot g^{(r)}(a^2) + \left( \frac{a^4}{2n^*} \right) \{ \text{Var}(N) + (E(N) - n^*)^2 \} \left\{ a^{-2} \left( \frac{r}{2} - 1 \right) - \frac{1}{2} \right\} \cdot g^{(r)}(a^2) + O(d^6d^{-3}) \]

\[ = \alpha + \left( \frac{a^2}{n^*} \right) \{ L - \eta^{-1}(s + 2q^{-1}) g^{(r)}(a^2) + \left( \frac{a^4}{2n^*} \right) \left( \frac{2n^*}{\eta q} \right) + 0(d^{-2}) + (L - \eta^{-1}(s + 2q^{-1}) \}

\[ + \left( \frac{a^4}{2n^*} \right) \left( \frac{2n^*}{\eta q} \right) + 0(d^{-2}) + (L - \eta^{-1}(s + 2q^{-1}) \}

\[ = \alpha + \left( \frac{a^2}{n^*} \right) \{ L - \eta^{-1}(s + 2q^{-1}) + (\eta q)^{-1} \left( \frac{r}{2} - 1 \right) - \frac{a^2}{2} \} g^{(r)}(a^2) + 0(d^4) + 0(d^2) + 0(d^3) \]

And the Result follows

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