# An Analysis of Properties in $L(OpenF_\sigma, Open)$ Functions in Topological Spaces

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**Abstract:** $D$**-Continuous mappings prompted us to study a new class of functions namely** $L(OpenF_\sigma, Open)$ **functions which contains properly the class of totally continuous mappings and is contained in the class ofcontinuous mappings. A few properties of these functions are discussed in this paper.**

***Keywords: Continuous mappings, Open*** $F_\sigma$ ***, Inverse image, Domain, Hausdorff space,*** $D$ ***-regular space.***

## 1. Introduction:

According to Noiri and Yuksel [1], [2] define various functions from a topological space $X$ to another topological space $Y$ have been introduced so far. These functions are continuous, non-continuous, weak continuous, strong continuous by various authors and researchers from time to time under different designations. A mapping $f : X \to Y$ is said to be $D$- continuous if inverse image of every open $F_\sigma$ set is open. Hamlett and Jankovic [5] showed that the collection ofall open $F_\sigma$ sets in a space constitutes a base for a weaker topology and both the topologies coincide if the space is $D$ –regular and hence continuous mappings with range space $D$- regular constitute a class which is the same as that of $D$- continuous mappings defined by Kohli [4] in the same reference, we study here, $L(OpenF_\sigma, Open)$ functions using the nomenclature for this we have to study $F_\sigma$, sets whichare the countable union of closed sets in a topological space. G. Aslim, A. Caksu Guler, and T. Noiri [6] study new functions are obviously a stronger form of continuity by means of open $F_\sigma$ sets which coincides with continuity if the domain space is a $D$- regular countable space. Actually in a $D$- regular space every open set is union of open $F_\sigma$ sets and count ability of the space gives it is countable union of open $F_\sigma$ sets and hence open $F_\sigma$ set it. Thus every open set is open $F_\sigma$ set in a $D$- regular countable space.

## 2. Definitions and Characterizations:

**Definition 2.1:**

A function $f : X \to Y$ is said to be $L(OpenF_\sigma, Open)$ at $x \in X$ if for each open set $V$ containing $f(x)$ there exists an open $F_\sigma$ set $U$ containing $x$, such that $F(U) \subset V$ and $f$ is called $L(OpenF_\sigma, Open)$ if it is $L(OpenF_\sigma, Open)$ at each $x$ in $X$

**Theorem2.1:** For a mapping $f : X \to Y$, the following are equivalent conditions,provided $X$ is a countable space.

a) $f$ is $L(OpenF_\sigma, Open)$

b) Inverse image of every member of a base is an open $F_\sigma$ set.

**Proof:** Let $V$ be a member of a base for $Y$, and $x \in f^{-1}(V)$ or, $x \in V$ so there exists an open $F_\sigma$ $F_\sigma$ set $U$ containing x such that $f(U) \subset V$, $x \in U \subset f^{-1}(V)$ thus $f^{-1}(V)$ is an open $F_\sigma$ set.
Hence $a \Rightarrow b$

There is the following porism according to above theorem:

a) Inverse image of every open set is an open $F_\sigma$ set.

b) Inverse image of every closed set is closed $G_\delta$ set.

c) For each $x \in X$ and each net $(x\alpha), \alpha \in D$ which is eventually, in eachopen $F_\sigma$ set containing , the net $f(x\alpha)$,

$\alpha \in D$ converges to $f(x)$

d) For each $x \in X$ and each filter base $\beta \in B_\lambda$, for which each open $F_\sigma$ set $V$ there exists $B_\lambda \in \beta$ such that $B_\lambda \subset V$, $f(\beta)$ converges to $f(x)$

**3. Comparison and analysis of the Properties:**

According to Yüksel, Açikgöz and Noiri [13] define that every cl-open set is Open $F_\sigma$, so every totally continuous [5]function is $L(OpenF_\sigma, Open)$ but the converse is not true.

1. The identity map on $(R, U)$, the real line is $L(OpenF_\sigma, Open)$ but not totally continuous defined by Hitir, and T. Noiri [10], [11] every $L(OpenF_\sigma, Open)$ mapping is continuous but the converse is not true as.

2. The identity map on $X = \{a,b\}$ with $T = \{\phi, \{a\}, \{b\}, \{a,b\}\}$ is super continuous [9] and hence continuous but not $L(OpenF_\sigma, Open)$

E. Hitir, A. Keskin, and T. Noiri [12] show that $L(OpenF_\sigma, Open)$ a map is independent of completely continuous mappings .and $\beta$ - continuous mappings.

3. Let $R$ be the usual space of real's and $Y = \{a,b,c,d\}$ with

$$T = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,b,c,d\}\}$$

Define a map $g: R \to Y$ by

$$g(x) = \begin{cases} a; x < p \\ b; p < x < q \\ c; q < x < r \\ d; x > r \end{cases}$$

Where $p, q \,\&\, r$ are distinct and real's, then $g$ is $L(OpenF_\sigma, Open)$ but not $\beta$ –continuous and hence not completely continuous asshown in [13] nor homeomorphism.

Let $X = \{a,b,c,d\}$ with $T = \{\phi, \{c\}, \{a,b\}, \{a,b,c,d\}\}$ and $Y = \{p,q,r\}$ with $U = \{\phi, \{p\}, Y\}$

Define a mapping $f: X \to Y$ by

$$f(x) = \begin{cases} f(a) = f(b) = p \\ f(c) = q \\ f(d) = r \end{cases}$$

Then $f$ is completely continuous, $\beta$ -continuous but not $L(OpenF_\sigma, Open)$ its shows that even a homeomorphism may fail to be $L(OpenF_\sigma, Open)$

**4. Basic Properties:**

For $L(OpenF_\sigma, Open)$ mappings the following theorem has some trivial results.

If $f: X \to Y$ and $g: Y \to Z$ is

1. Continuous whenever $f$ is $D$–continuous and $g$ is $L(OpenF_\sigma, Open)$ mapping.
2. $L(OpenF_\sigma, Open)$ Mapping whenever f is $L(OpenF_\sigma, Open)$ and $g$ is continuous.
3. Composition of two $L(OpenF_\sigma, Open)$ mappings is $L(OpenF_\sigma, Open)$ mapping.
4. $P_\alpha \quad f: X \to X_\alpha$ is $L$ (Open $F_\sigma$, Open) iff $f$ is $L(OpenF_\sigma, Open)$ where $P_\alpha$ is $\alpha^{th}$ projection of the product space onto $X_\alpha$.

**Theorem 4.1** If $f: X \to Y$ is a surjection carrying open $F_\sigma$ (or closed $G_\delta$) sets onto open (closed) sets and, $g: Y \to Z$ is any mapping such that $gof$ is $L(OpenF_\sigma, Open)$ then $g$ is continuous.

**Proof:** If $U$ is an open (or Closed) set in $Z$,
Then, $(gof)^{-1}U = f^{-1}\{g^{-1}(U)\}$ is open $F_\sigma$ (or closed $G_\delta$) set in $x$ and,
Hence $f^{-1}[f^{-1}\{g^{-1}(U)\}] = g^{-1}(U)$ is open (closed) in $Y$.

**Corollary 4.1(1):** Let $f: X \to Y$ be $L$ (Open $F_\sigma$, Open) and $A \subset X$
If $G$ is open in $Y$, then $f^{-1}(G)$ is open - $F_\sigma$ in $X$.
Al-shami, El-Shafei, and Abo-Elhamayel [3] shows that $f^{-1}(G)$ a countable union of closed sets in $X$ and,
Hence, intersection of $f^{-1}(G)$ and $A = f\{A^{-1}(G)\}$ is a countable union too of closed sets in $A$. Thus $f(A): A \to Y$ is $L(OpenF_\sigma, Open)$.

**Condition 4.1(2):** If $A$ is open $F_\sigma$ in $B$ and $B$ is open $F_\sigma$ in $X$ then $A$ is open $F_\sigma$ in $X$.
**Explanation:** Since $A$ is $F_\sigma$ in $B$ so $A = UB_i$ where each $B_i$ is closed in $B$.
Also, $B_i = X_i$ intersection $B$ where each $X_i$ is closed in $X$.
Thus, $A$ = union of $X_i$ intersection, $B = \cup X_i$ intersection $B$ is a $F_\sigma$ set in $X$ being intersection of two $F_\sigma$ sets.Openness of $A$ is obvious in $X$.

Noiri and Umehara [9] notified that if $A$ is closed $G_\delta$ set in $B$ and $B$ is closed $G_\delta$ set in $X$ then $A$ is closed $G_\delta$ set in $X$.

**Theorem 4.2** Let $X$ = $A$ union $B$, where $A$ and $B$ are open $F_\sigma$ sets in $X$, and $f: A \to Y$, $g: B \to Y$ be $L$ (Open $F_\sigma$, Open) functions.
If $f(x) = g(x)$ for every $x$ in $A \cap B$, then $h: X \to Y$, defined by

$$h(x) = \begin{cases} f(x) if x \in A \\ g(x) if x \in B \end{cases}$$

is $L(OpenF_\sigma, Open)$.

**Proof:** Since $h^{-1}(U) = f^{-1}(U)$ union $g^{-1}(U)$ for every open set $U$ in $Y$, therefore $h^{-1}(U)$ is open is open $F_\sigma$ as $f^{-1}(U)$ and $g^{-1}(U)$ are both open $F_\sigma$ in open $F_\sigma$ sets $A$ and $B$ respectively and hence the same in $X$.

**Corollary 4.2(1)** If $A$ and $B$ are closed $G_\delta$ sets in $X$, in instead of open $F_\sigma$ sets, the theorem remains unaltered.
**Corollary 4.2(2)** If $X$ = $U$ $(U_\alpha : \alpha \in A)$ where $U_\alpha$ are open $F_\sigma$ and pairwise disjointsets in $X$, such that $f_\alpha : U_\alpha \to Y$ is $L(OpenF_\sigma, Open)$ for each $\alpha$.
Then $h: X \to Y$ defined by $h(x) = f_\alpha(x)$ if $x \in U_\alpha$ is a $L(OpenF_\sigma, Open)$ Function.

**Theorem 4.3** Let $f$ and $g$ be $L(OpenF_\sigma, Open)$ function from a space $X$ into a $T_2$-space $Y$ then,
$A = \{x : f(x) = g(x)\}$ Is closed $G_\delta$ in $X$, provided $A$ is Co-countable.
**Proof:** For each $x \in X - A$, we can show that the existence of open $F_\sigma$ set $G : x \in G \subset X - A$.
Thus, $X - A$ is countable union of $F_\sigma$ sets and hence, $A$ is closed $G_\delta$.
Condition: If $f$ and $g$ agree on a co-countable set $B$ such that smallest closed $G_\delta$ set containing $B$ is $X$ Then, $f = g$.

A space $X$ is called $D$-regular [14] if for each $x \in X$ and each open set $U$ containing $x$, there exists an open $F_\sigma$ set $V$ such that

$x \in V \subset U$. Obviously, every open set in a $D$-regular space is union of open $F_\sigma$ sets and hence, will be open $F_\sigma$ itself provided it is countable.

**Corollary 4.3(1)** Let $(X,T)$ be a topological space, the following statements are equivalent

**a.** $(X,T)$ is a $D$-regular and countable space ($i.e.$, set $X$ is countable)

**b.** Every continuous functions $f$ from $X$ into a topological space $Y$ in $L(OpenF_\sigma, Open)$ function

**Corollary 4.3(2)** The set of fixed points of an $L(OpenF_\sigma, Open)$ function on a Hausdorff defined by Ajmal and Kohli [8],[7] and $D$-regular and countable space $X$ is a closed $G_\delta$ set, where $A = \{x : x \in f(x)\}$ Identity map on $X$ is obviously $L$ (Open $F_\sigma$, Open) in view of Continuous functions.

Thus $f$ and identity map are fulfilling so $A$ is closed $G_\delta$ set.

## 5 Conclusion and Future Work:

Every open set is $F_\sigma$ in a perfectly normal space and a space is perfect if every open (or closed) set is $F_\sigma$ (or $G_\delta$ set),so every continuous function from a perfect space is $L(OpenF_\sigma, Open)$ introduced a function $f : X \to Y$ to be $z-$continuous if for each $x \in X$ and each co-zero set $V$ containing $f(x)$, there is an open set $U$ containing $x$ such that $f(U) \subset V$ for every $D$-Continuous function is $z-$continuous but the converse fails. We have that every $z-$continuous function is a $D$-Continuous provider where the range space is normal. Since in a normal space every closed $G_\delta$ set is a zero set or every open $F_\sigma$ set is a co-zero set. The following conclusion is trivial:

*i.* If $X$ is a countable set and *zero* –dimensional space then every continuous mapping from $X$ $L(OpenF_\sigma, Open)$

*ii.* For each $\alpha \in I$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a mapping and let $f_\alpha : \pi X_\alpha \to \pi Y_\alpha$ be defined as $f(x\alpha) = f_\alpha(x_\alpha)$ for each $(x\alpha)$ in $\pi X_\alpha$.

**Conflict of Interest:**

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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