On Hyperbolic Hsu-Structure Manifold, Birecurrent and Symmetry

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Abstract- **In this paper, we have defined bi recurrence and symmetry of different kinds in H- Hsu-structure manifold. Some theorem establishing relationship between different kinds of birecurrent H- HSU-Structure manifold involving equivalent conditions with respect to projective, conformal, conharmonic and concircular curvature tensors has been discussed recurrence parameter have also been studied.**

Index Terms- **BiRecurrence parameter, Curvature Tensors,C^{∞} -function, Hsu-structure manifold.**

1.INTRODUCTION

If a differentiable manifold Vn, of differentiability class C^{∞}.there be in Vn,a vector valued linear function F of classC^{∞}, satisfying the algebraic equation

$\bar{\bar{X}}$ = - a$^r X$, for arbitrary vector field X . (1.1)

Where $\bar{X}$= FX , 0≤ $r \leq n$ and 'a'is real or imaginary number,then{F} is said to give to Vn a Hyperbolic HSU-structure defined by the equations(1.1) and the manifold Vn is called a Hyperbolic HSU –structure manifold. Hyperbolic HSU-structure manifold or briefly H-Hsu-structure manifold.

Remark(1.1) : The equation (1.1) gives different structures for different values of 'a' and r .

If a= ± 1 and r =2 ,it is an almost complex structure.
If a= $\pm i$ and r = 2 , it is an almost product structure or a hyperbolic almost complex structure.
If a = 0, it an almost tangent structure or almost hyperbolic tangent structure.
If a≠0, s It is the hyperbolic π-structure.

Let the Hsu – structure Vn, be endowed with a Hermitian metric tensor g, such that
g($\bar{X}$,$\bar{Y}$) –a^r g(X,Y) =0,
Then{F,g} is said to give Vn a hyperbolic Hsu-structure metric manifold.
Agreement (1.1):In what follows and the above, the equations containing X,Y,Z………..,etc. hold for these arbitrary vector in Vn,
The curvature tensor K,a vector –valued tri-linear function w.r.t the connexion D is given by

$$K(X,Y,Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z, \qquad (1.2)a$$

Where

$$[X,Y] = D_X Y - D_Y X \qquad (1.2)b$$

The Ricci tensor in $V_n$is given by

$$\text{Ric}(Y,Z) = (C_1^1 K)(Y,Z). \qquad (1.3)$$

Where by ($C_1^1 K)(Y,Z)$, we mean the contraction of $K(X,Y,Z)$with respect the first slot.
For Ricci tensor, we also have

$$Ric(Y,Z) = Ric(Z,Y), \qquad (1.4)a$$

$$Ric(Y,Z) = g(r(Y),Z) = g\big(Y, r(Z)\big), \qquad (1.4)b$$

$$(C_1^1 r) = R \qquad (1.4)c$$

Let W,C,L and V be the Projective,Conformal, conharmonic and concircular curvature tensors respectively given by

$W(X,Y,Z) = K(X,Y,Z) - \frac{1}{(n-1)}[Ric(Y,Z)X - Ric(X,Z)Y]$ (1.5)

$C(X,Y,Z) = -\frac{1}{(n-2)}\{Ric(Y,Z)X - Ric(X,Z)Y - g(X,Z)r(Y) + g(Y,Z)r(X)\} + \frac{R}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]$. (1.6)

$L(X,Y,Z) = K(X,Y,Z) - \frac{1}{(n-2)}[Ric(Y,Z)X - Ric(X,Z)Y - g(X,Z)r(Y) + g(Y,Z)r(X)]$. (1.7)

$V(X,Y,Z) = K(X,Y,Z) - \frac{R}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]$. (1.8)

A manifold is said to be recurrent, if

$(\nabla K)(X,Y,Z,T) = A(T_1)K(X,Y,Z)$.

The recurrent manifold is said to be symmetric, if
$A(T_{1)} = 0$, in the equation (1.9).

II. BIRECURRENCE AND SYMMETERY OF DIFFERENT KINDS

Let Q, a vector – valued trilinear function be any one of the curvature tensors K,W,C,L or V. Then we will define recurrence of different kinds as follows:

Definition(2.1). A –HSU-structure manifold is said to be (1)-birecurrent in Q,if
$a^r(\nabla\nabla Q)(X,Y,Z,T,S) - (\nabla Q)\big((\nabla F)(\bar{X},T),Y,Z,S\big) - (\nabla Q)\big((\nabla F)(\bar{X},S),Y,Z,T\big) - Q((\nabla\nabla F)(\bar{X},T,S),Y,Z) = a^r P_1(T,S)Q(X,Y,Z)$ (2.1)

Where P_2(T,S) is non – vanishing C^∞,called birecurrence parameter.

Definition(2.2). A H-HSU- structure manifold is said to be Ricci (1)-birecurrent ,if
$a^r(\nabla\nabla \mathrm{Ric})(Y,Z,T,S) - (\nabla \mathrm{Ric})(\big((\nabla F)(\bar{Y},T),Z,S\big) - (\nabla Ric)\ \big((\nabla F)(\bar{Y},S),Z,T\ \big) - Q((\nabla\nabla Ric)(\bar{Y},T,S),Z) = a^r P_2(T,S)Ric(Y,Z)$. (2.2)

Definition(2.3). A H-HSU- structure manifold is said to be- (1,2)-birecurrent inQ,if $a^r(\nabla\nabla Q)(X,\bar{Y},Z,T,S) - (\nabla Q)\big((\nabla F)(\bar{X},T),\bar{Y},Z,S\big) - (\nabla Q)\big((\nabla F)(\bar{X},S),\bar{Y},Z,T\big) + a^r(\nabla Q)(X,(\nabla F)(Y,T),Z,S) + a^r(\nabla Q)(X,(\nabla F)(Y,S),Z,T) - Q\big((\nabla F)(\bar{X},T),(\nabla F)(Y,S),Z\big) - Q\big((\nabla F)(\bar{X},S),(\nabla F)Y,T\big),Z) - \mathrm{Q}\big((\nabla\nabla \mathrm{F})(\bar{\mathrm{X}},T,S),\bar{Y},Z\big) + a^r Q(X,(\nabla\nabla F)(Y,T,S),Z) = a^r P_2(T,S)Q(X,\bar{Y},Z)$, (2.3)

Definition(2.4). A H-HSU- structure manifold is said to be Ricci- (1,2)-bi recurrent ,if
$a^r(\nabla\nabla \mathrm{Ric})(,Y,\bar{Z},T,S) - (\nabla Ric)\big((\nabla F)(\bar{Y},T),\bar{Z},S\big) - (\nabla Ric)\big((\nabla F)(\bar{Y},S),\bar{Z},T\big) + a^r(\nabla Ric)(Y,(\nabla F)(Z,T),S) + a^r(\nabla Ric)(Y,(\nabla F)(Z,S),T) - Ric\big((\nabla F)(\bar{Y},T),(\nabla F)(Z,S)\big) - Ric\big((\nabla F)(\bar{Y},S),(\nabla F)(Z,T)\big) - Ric\big((\nabla\nabla F)(\bar{Y},T,S),\bar{Z}\big) + a^r Ric(Y,(\nabla\nabla F)(Z,T,S)) = a^r P_2(T,S)Ric(Y,\bar{Z})$. (2.4)

Definition(2.5). A H-HSU- structure manifold is said to be- (1,2,3)-birecurrent ,if

$a^r(\nabla\nabla Q)(X,\bar{Y},\bar{Z},T,S) - (\nabla Q)\big((\nabla F)(\bar{X},T),\bar{Y},\bar{Z},S\big) - (\nabla Q)\big((\nabla F)(\bar{X},S),\bar{Y},\bar{Z},T\big) + a^r(\nabla Q)(X,(\nabla F)(Y,T),\bar{Z},S) + a^r(\nabla Q)(X,(\nabla F)(Y,S),\bar{Z},T) + a^r(\nabla Q)(X,\bar{Y},(\nabla F)(Z,T),S) + a^r(\nabla Q)(X,\bar{Y},(\nabla F)(Z,S),T) - Q\big((\nabla F)(\bar{X},T),(\nabla F)(Y,S),\bar{Z}\big) - Q\big((\nabla F)(\bar{X},S),(\nabla F)Y,T\big),\bar{Z}) + a^r Q(X,(\nabla F)(Y,T),(\nabla F)(Z,S) + a^r Q((\nabla F)(X,S),Y(\nabla F)(Z,T) + a^r Q((\nabla\nabla F)(X,T,S),Y,\bar{Z}) - \mathrm{Q}(\bar{\mathrm{X}},(\nabla\nabla \mathrm{F})(\bar{\mathrm{Y}},T,S),\bar{Z}) + a^r Q(\bar{X},Y,(\nabla\nabla F)(Z,T,S)) = a^r P_2(T,S)Q(\bar{X},Y,Z)$. (2.5)

Similarly,(2),(3),(4),(2,3),(2,4),(1,3),(1,4),(3,4),(1,2,4),(1,3,4),(2,3,4) and (1,2,3,4) birecurrence in Q can also be defined.

Definition(2.6). A(1),(1,2) and (1,2,3)- birecurrent H-HSU-structure manifold is said to be Q-symmetric or Ricci-symmetric, if

$A_2(T,S) = 0,$ (2.6)

In the above equations.

Theorem (2.1) A Q-(1,2) H-HSU-structure manifold is Q-(1)-birecurrent for the same recurrence parameter , provided.

$$a^r(\nabla Q)(X,(\nabla F)(Y,T),Z,S)+a^r(\nabla Q)(X,(\nabla F)(Y,S),Z,T)-Q\big((\nabla F)(\bar{X},T),(\nabla F)(Y,S),Z\big)$$
$$-Q\big((\nabla F)(\bar{X},T),(\nabla F)(Y,S),Z\big)-Q\big((\nabla F)(\bar{X},S),(\nabla F)(Y,T),Z\big)+a^r Q(X,(\nabla\nabla F)(Y,T,S,),Z)$$
$$=0 \qquad (2.7)$$

Proof. Let the manifold is Q-(1)-birecurrent then barring Y in equation (2.1),we get

$$a^r(\nabla\nabla Q)(X,Y,Z,T,S)-(\nabla Q)\big((\nabla F)(\bar{X},T),Y,Z,S\big)-(\nabla Q)\big((\nabla F)(\bar{X},S),Y,Z,T\big)$$
$$-Q\big((\nabla\nabla F)(\bar{X},T,S),Y,Z\big)=a^r P_2(T,S)Q(X,\bar{Y},Z)\,, \qquad (2.8)$$

Now assuming that a Q-(1,2)-birecurrent H-HUS-structure manifold is Q-(1)-birecurrent then comparing (2.3) and (2.8), we get equation (2.7)

Theorem(2.2) A Q-(1,2) H-HSU-structure manifold is Q-(2) –birecurrent for the same recurrence parameter , provided.

$$a^r(\nabla Q)\big((\nabla F)(X,T),Y,Z,S\big)+a^r(\nabla Q)\big((\nabla F)(X,S),Y,Z,T\big)-Q\;(\nabla F)(X,T),(\nabla F)(\bar{Y},S),Z)$$
$$-Q\big((\nabla F)(X,S),(\nabla F)(\bar{Y},T),Z\big)+a^r Q\big((\nabla\nabla F)(X,T,S,),Y,Z\big)=0 \qquad (2.9)$$

Proof. Let the manifold is Q-(2)-birecurrent then barring X in Q-(2) birecurrent manifold, such that.

$$a^r(\nabla\nabla Q)(\bar{X},Y,Z,T,S)-(\nabla Q)(\bar{X},(\nabla F)(\bar{Y},T),Z,S)-(\nabla Q)(\bar{X},(\nabla F)(\bar{Y},S),Z,T)$$
$$-Q(\bar{X},(\nabla\nabla F)(\bar{Y},T,S),Z)=a^r P_2(T,S)Q(\bar{X},Y,Z)\,. \qquad (2.10)$$

Now assuming that a Q-(1,2)- birecurrent H-HSU-structure manifold is Q-(2)-birecurrent then comparing equation (2.3) and (2.10), we get equation (2.9).

Theorem (2.3) . In a (1)- birecurrent H-HSU-structure manifold if any two of the following conditions hold for the same recurrence parameter then third also hold:

(a). It is conformal –(1)- birecurrent ,
(b). It is conharmonic-(1)-birecurrent,
(c). It is concircular –(1)- birecurrent,

Proof (2.3). We have

$$C(\bar{X},Y,Z)=L(\bar{X},Y,Z)+\frac{n}{(n-2)}[K(\bar{X},Y,Z)-V(\bar{X},Y,Z)]. \qquad (2.11)$$

Now, from equation from from (1.1) and (2.11),we have

$$a^r P_2(T,S)C(X,Y,Z)=a^r P_2(T,S)L(X,Y,Z)+\frac{na^r}{(n-2)}P_2(T,S)\{K(X,Y,Z)-V(X,Y,Z)\}, \qquad (2.12)$$

Differentiating equation (2.11) with respect to T and S successively , using equation (2.11) and then barring X in the equation , we get

$$-a^r(\nabla\nabla C)(X,Y,Z,T,S)+(\nabla C)\big((\nabla F)(\bar{X},T),Y,Z,S\big)$$
$$+(\nabla C)(\nabla F)(\bar{X},S),Y,Z,T)\;+C((\nabla\nabla F(\bar{X},T,S),Y,Z)$$
$$=-a^r(\nabla\nabla L)(X,Y,Z,T,S)\;+(\nabla L)\big((\nabla F)(\bar{X},T),Y,Z,S\big)+(\nabla L)\big((\nabla F)(\bar{X},S),Y,Z,T\big)+L((\nabla\nabla F)(\bar{X},T,S),Y,Z)+$$
$$\frac{n}{(n-1)}\{-a^r(\nabla\nabla K)(X,Y,Z,T,S)+(\nabla K)((\nabla F)(\bar{X},T),Y,Z,S)+$$
$$(\nabla K)((\nabla F)(\bar{X},S),Y,Z,T)+K((\nabla\nabla F)(\bar{X},T,S),Y,Z)+a^r(\nabla\nabla V)(X,Y,Z,T,S)-(\nabla V)\big((\nabla F)(\bar{X},T),Y,Z,S\big)-$$
$$(\nabla V)\big((\nabla F)(\bar{X},S),Y,Z,T\big)-V\big((\nabla\nabla F)(\bar{X},T,S),Y,Z\big) \qquad (2.13)$$

Adding equation (2.12) from equation (2.13), we get

$$-a^r(\nabla\nabla C)(X,Y,Z,T,S)+(\nabla C)\big((\nabla F)(\bar{X},T),Y,Z,S\big)$$
$$+(\nabla C)(\nabla F)(\bar{X},S),Y,Z,T)\;+C((\nabla\nabla F(\bar{X},T,S),Y,Z)+a^r P_2(T,S)C(X,Y,Z)$$

$= -a^r(\nabla\nabla L)(X,Y,Z,T,S) \quad + (\nabla L)\big((\nabla F)(\bar{X},T),Y,Z,S\big) + (\nabla L)\big((\nabla F)(\bar{X},S),Y,Z,T\big) +$
$L((\nabla\nabla F)(\bar{X},T,S),Y,Z) + a^r P_2(T,S)L(X,Y,Z) + \frac{n}{(n-1)}\{-a^r(\nabla\nabla K)(X,Y,Z,T,S) + (\nabla K)((\nabla F)(\bar{X},T),Y,Z,S) +$
$(\nabla K)((\nabla F)(\bar{X},S),Y,Z,T) + K((\nabla\nabla F)(\bar{X},T,S),Y,Z) +$
$a^r P_2(T,S)\{K(X,Y,Z) + a^r(\nabla\nabla V)(X,Y,Z,T,S) - (\nabla V)\big((\nabla F)(\bar{X},T),Y,Z,S\big) - (\nabla V)\big((\nabla F)(\bar{X},S),Y,Z,T\big) -$
$V\big((\nabla\nabla F)(\bar{X},T,S),Y,Z\big) - a^r P_2(T,S)V(X,Y,Z)\}.$ (2.14)

If a (1)-birecurrent H-HSU-structure manifold is conformal-(1)- birecurrent and conharmonic-(1)-birecurrent for the same recurrence parameter then from equation (2.14),we have

$a^r(\nabla\nabla V)(X,Y,Z,T,S) - (\nabla V)\big((\nabla F)(\bar{X},T),Y,Z,S\big)$
$-(\nabla V)\big((\nabla F)(\bar{X},S),Y,Z,T\big) - V\big((\nabla\nabla F)(\bar{X},T,S),Y,Z\big) = a^r P_2(T,S)V(X,Y,Z)$

Similarly ,it can be shown that if the manifold is either conformal-(1)-birecurrent and concircular –(1) –birecurrent then it is either conharmonic –(1)- birecurrent and concircular-(1)-birecurrent then it is either conharmonic-(1)-birecrrent then it is either conharmonic-(1)-birecurrent or conformal-(1)-birecurrent for the same recurrence parameter.

Theorem (2.4). . In a (1,2)- birecurrent H-HSU-structure manifold if any two of the following conditions hold for the same recurrence parameter then third also hold:

(a). It is conformal –(1,2)- birecurrent ,
(b). It is conharmonic-(1,2)-birecurrent,
(c). It is concircular –(1,2)- birecurrent,

Proof. Barring X and Y in equation (2.11), we get

$C(\bar{X},\bar{Y},Z) = L(\bar{X},\bar{Y},Z) + \frac{n}{(n-2)}[K(\bar{X},\bar{Y},Z) - V(\bar{X},\bar{Y},Z)]$ (2.15)

Now, from equation from from (1.1) and (2.15),we have

$a^r P_2(T,S)L(X,\bar{Y},Z) = a^r P_2(T,S)L(X,\bar{Y},Z) + \frac{na^r}{(n-2)}P_2(T,S)\{K(\bar{X},\bar{Y},Z) - V(\bar{X},\bar{Y},Z)\}$ (2.16)

Differentiating equation (2.15) with respect to T and S successively , using equation (2.15) and then barring X in the equation , we get

$-a^r(\nabla\nabla C)(X,\bar{Y},Z,T,S) + (\nabla C)\big((\nabla F)(\bar{X},T),\bar{Y},Z,S\big) + (\nabla C)(\nabla F)(\bar{X},S),\bar{Y},Z,T)$
$-a^r(\nabla C)(X,(\nabla F(Y,T),Z,S) - a^r(\nabla C)(X,(\nabla F(Y,S),Z,T) + C\big((\nabla F)(\bar{X},T),(\nabla F)(Y,S),Z\big) + C\big((\nabla F)(\bar{X},S),(\nabla F)(Y,T),Z\big) +$
$C((\nabla\nabla F(\bar{X},T,S),\bar{Y},Z) - a^r C(X,\nabla\nabla F)(Y,T,S),Z = \; -a^r(\nabla\nabla L)(X,\bar{Y},Z,T,S) + (\nabla L)\big((\nabla F)(\bar{X},T),\bar{Y},Z,S\big) +$
$(\nabla L)\big((\nabla F)(\bar{X},S),\bar{Y},Z,T\big) - a^r(\nabla L)(X,(\nabla F)(Y,Z),Z,S) - a^r(\nabla L)(X,(\nabla F)(Y,S),Z,T)$
$+L((\nabla F)(\bar{X},T),(\nabla F)(Y,S),Z) + L((\nabla F)(\bar{X},S),(\nabla F)(Y,T),Z) + L((\nabla\nabla F)(\bar{X},T,S),(\bar{Y},,Z) -$
$a^r L(X,(\nabla\nabla F)(Y,T,S),Z) + \frac{n}{(n-2)}\{-a^r(\nabla\nabla K)(X,\bar{Y},Z,T,S) + (\nabla K)(\nabla F)(\bar{X},T),\bar{Y},Z,S)$
$(\nabla K)(\nabla F)(\bar{X},S),\bar{Y},Z,T) - a^r(\nabla K)(X,(\nabla F)(Y,T),Z,S) - a^r(\nabla K)(X,(\nabla F)(Y,S),Z,T)$
$+K\big((\nabla F)(\bar{X},T),(\nabla F)(Y,S),Z\big) + K\big((\nabla F)(\bar{X},S),(\nabla F)(Y,T),Z\big) + K((\nabla\nabla F)(\bar{X},T,S),\bar{Y},Z)$
$-a^r K((X,(\nabla\nabla F)(Y,T,S),Z) + a^r(\nabla\nabla V)(X,\bar{Y},Z,T,S) - (\nabla V)((\nabla F)(\bar{X},T),\bar{Y},Z,S)$
$-(\nabla V)((\nabla F)(\bar{X},S),\bar{Y},Z,T) + a^r(\nabla V)(X,(\nabla F)(Y,T),Z,S) + a^r(\nabla V)(X,(\nabla F)(Y,S),Z,T)$
$-V((\nabla F)(\bar{X},T),(\nabla F)(Y,S),Z) - V((\nabla F)(\bar{X},S),(\nabla F)(Y,T),Z) - V((\nabla\nabla F)(\bar{X},T,S),\bar{Y},Z)$
$-a^r V(X,(\nabla\nabla F)(Y,T,S),Z)\}.$ (2.17)

Adding equation (2.16) from equation (2.17), we get,

$-a^r(\nabla\nabla C)(X,\bar{Y},Z,T,S) + (\nabla C)\big((\nabla F)(\bar{X},T),\bar{Y},Z,S\big) + (\nabla C)(\nabla F)(\bar{X},S),\bar{Y},Z,T)$
$-a^r(\nabla C)(X,(\nabla F(Y,T),Z,S) - a^r(\nabla C)(X,(\nabla F(Y,S),Z,T) + C\big((\nabla F)(\bar{X},T),(\nabla F)(Y,S),Z\big) + C\big((\nabla F)(\bar{X},S),(\nabla F)(Y,T),Z\big) +$
$C((\nabla\nabla F(\bar{X},T,S),\bar{Y},Z) - a^r C(X,\nabla\nabla F)(Y,T,S),Z + a^r P_2(T,S)L(X,\bar{Y},Z) = -a^r(\nabla\nabla L)(X,\bar{Y},Z,T,S) +$
$(\nabla L)\big((\nabla F)(\bar{X},T),\bar{Y},Z,S\big) + (\nabla L)\big((\nabla F)(\bar{X},S),\bar{Y},Z,T\big) - a^r(\nabla L)(X,(\nabla F)(Y,S),Z,T) + L((\nabla F)(\bar{X},T),(\nabla F)(Y,S),Z)$
$+ L((\nabla F)(\bar{X},S),(\nabla F)(Y,T),(\nabla L)(X,(\nabla F)(Y,Z),Z,S)Z)$
$+ L((\nabla\nabla F)(\bar{X},T,S),(\bar{Y},,Z) - a^r L(X,(\nabla\nabla F)(Y,T,S),Z) + a^r P_2(T,S)L(X,\bar{Y},Z)$
$+\frac{n}{(n-2)}\{-a^r(\nabla\nabla K)(X,\bar{Y},Z,T,S) + (\nabla K)(\nabla F)(\bar{X},T),\bar{Y},Z,S)$
$(\nabla K)(\nabla F)(\bar{X},S),\bar{Y},Z,T) - a^r(\nabla K)(X,(\nabla F)(Y,T),Z,S) - a^r(\nabla K)(X,(\nabla F)(Y,S),Z,T)$
$+K\big((\nabla F)(\bar{X},T),(\nabla F)(Y,S),Z\big) + K\big((\nabla F)(\bar{X},S),(\nabla F)(Y,T),Z\big) + K((\nabla\nabla F)(\bar{X},T,S),\bar{Y},Z)$

$+a^r P_2(T,S)K(X,\bar{Y},Z) - a^r K((X,(\nabla\nabla F)(Y,T,S),Z) + a^r(\nabla\nabla V)(X,\bar{Y},Z,T,S) - (\nabla V)((\nabla F)(\bar{X},T),\bar{Y},Z,S)$
$-(\nabla V)((\nabla F)(\bar{X},S),\bar{Y},Z,T) + a^r(\nabla V)(X,(\nabla F)(Y,T),Z,S) + a^r(\nabla V)(X,(\nabla F)(Y,S),Z,T) - V((\nabla F)(\bar{X},T),(\nabla F)(Y,S),Z) -$
$V((\nabla F)(\bar{X},S),(\nabla F)(Y,T),Z)$
$-V((\nabla\nabla F)(\bar{X},T,S),\bar{Y},Z) + a^r V(X,(\nabla\nabla F)(Y,T,S),Z) - a^r P_2(T,S)V(X,\bar{Y},Z)\}.$ (2.18)

If a (1,2) –birecurrent H-HSU –structure manifold is conformal-(1,2)-birecurrent and conharmonic-(1,2)-birecurrent then from equation (2.18)a,we have

$a^r(\nabla\nabla V)(X,\bar{Y},Z,T,S) - (\nabla V)((\nabla F)(\bar{X},T),\bar{Y},Z,S) - (\nabla V)((\nabla F)(\bar{X},S),\bar{Y},Z,T) + a^r(\nabla V)(X,(\nabla F)(Y,T),Z,S)$
$+ a^r(\nabla V)(X,(\nabla F)(Y,S),Z,T) - V((\nabla F)(\bar{X},T),(\nabla F)(Y,S),Z) - V((\nabla F)(\bar{X},S),(\nabla F)(Y,T),Z)$
$-V((\nabla\nabla F)(\bar{X},T,S),\bar{Y},Z) + a^r V(X,(\nabla\nabla F)(Y,T,S),Z) = a^r P_2(T,S)V(X,\bar{Y},Z)\}.$

Similarly ,it can be shown that if the manifold is either conformal-(1,2)-birecurrent and concircular –(1,2) –birecurrent then it is either conharmonic –(1,2)- birecurrent and concircular-(1,2)-birecurrent then it is either conharmonic-(1,2)-birecrrent then it is either conharmonic-(1,2)-birecurrent or conformal-(1,2)-birecurrent for the same recurrence parameter.

Theorem (2.5). . In a (1,2,3)- birecurrent H-HSU-structure manifold if any two of the following conditions hold for the same recurrence parameter then third also hold:

(a). It is conformal –(1,2,3)- birecurrent ,
(b). It is conharmonic-(1,2,3)-birecurrent,
(c). It is concircular –(1,2,3)- birecurrent,

Proof. Barring X , Y and Z in equation (2.11), we get

$C(\bar{X},\bar{Y},\bar{Z}) = L(\bar{X},\bar{Y},\bar{Z}) + \frac{n}{(n-2)}[K(\bar{X},\bar{Y},\bar{Z}) - V(\bar{X},\bar{Y},\bar{Z})]$ (2.18)b

Now, from equation from from (1.1) and (2.18),we have

$a^r P_2(T,S)L(X,\bar{Y},\bar{Z}) = a^r P_2(T,S)L(X,\bar{Y},\bar{Z}) + \frac{na^r}{(n-2)}P_2(T,S)\{K(X,\bar{Y},\bar{Z}) - V(X,\bar{Y},\bar{Z})\}$ (2.19)

Differentiating equation (2.18)b with respect to T and S successively , using equation (2.18)b and then barring X and using (1.1) in the resulting equation , we get

$-a^r(\nabla\nabla C)(X,\bar{Y},\bar{Z},T,S) + (\nabla C)((\nabla F)(\bar{X},T),\bar{Y},\bar{Z},S) + (\nabla C)(\nabla F)(\bar{X},S),\bar{Y},\bar{Z},T)$
$-a^r(\nabla C)(X,(\nabla F(Y,T),\bar{Z},S) - a^r(\nabla C)(X,(\nabla F(Y,S),\bar{Z},T) - a^r(\nabla C)(X,\bar{Y}(\nabla F(Z,S),T)$
$-a^r(\nabla C)(X,\bar{Y}(\nabla F(Z,T),S) - a^r C(X,(\nabla F(Y,S),(\nabla F)(Z,T)$
$-a^r C(X,(\nabla F(Y,T),(\nabla F)(Z,S) + C((\nabla F(\bar{X},T),\bar{Y},(\nabla F)(Z,S)$
$+C((\nabla F(\bar{X},S),\bar{Y},(\nabla F)(Z,T) + C((\nabla F(\bar{X},T),(\nabla F)(Y,S),\bar{Z})$
$+C((\nabla F(\bar{X},S),(\nabla F)(Y,T),\bar{Z}) + C((\nabla F)(\bar{X},T),(\nabla F)(Y,S),Z)$
$+C((\nabla F)(\bar{X},S),(\nabla F)(Y,T),Z) + C((\nabla\nabla F(\bar{X},T,S),\bar{Y},Z) - a^r C(X,\nabla\nabla F)(Y,T,S),Z$
$+C((\nabla\nabla F(\bar{X},T,S),\bar{Y},\bar{Z}) - a^r C(X,\nabla\nabla F)(Y,T,S),\bar{Z}) - a^r C(X,\bar{Y}(\nabla\nabla F)(Z,T,S)$
$= -a^r(\nabla\nabla L)(X,\bar{Y},\bar{Z},T,S) + (\nabla L)((\nabla F)(\bar{X},T),\bar{Y},\bar{Z},S) + (\nabla L)((\nabla F)(\bar{X},S),\bar{Y},\bar{Z},T)$
$-a^r(\nabla L)(X,(\nabla F)(Y,T),\bar{Z},S) - a^r(\nabla L)(X,(\nabla F)(Y,S),\bar{Z},T) - a^r(\nabla L)(X,\bar{Y}(\nabla F)(Z,S),T)$
$-a^r(\nabla L)(X,\bar{Y}(\nabla F)(Z,T),S) - a^r L(X,(\nabla F)(Y,S),(\nabla F)(Z,T) - a^r L(X,(\nabla F)(Y,T),(\nabla F)(Z,S)$

$+L((\nabla F)(\bar{X},T),\bar{Y},(\nabla F)(Z,S)) + L((\nabla F)(\bar{X},S),\bar{Y},(\nabla F)(Z,T)) + L((\nabla F)(\bar{X},T),(\nabla F)(Y,S),\bar{Z})$
$+L((\nabla F)(\bar{X},S),(\nabla F)(Y,T),\bar{Z}) + L((\nabla\nabla F)(\bar{X},T,S),(\bar{Y},,\bar{Z}) - a^r L(X,(\nabla\nabla F)(Y,T,S),\bar{Z})$
$-a^r L(X,\bar{Y}(\nabla\nabla F)(Z,T,S) + \frac{n}{(n-2)}\{-a^r(\nabla\nabla K)(X,\bar{Y},\bar{Z},T,S) + (\nabla K)(\nabla F)(\bar{X},T),\bar{Y},\bar{Z},S)$
$+(\nabla K)(\nabla F)(\bar{X},S),\bar{Y},\bar{Z},T) - a^r(\nabla K)(X,(\nabla F)(Y,T),Z,S) - a^r(\nabla K)(X,(\nabla F)(Y,S),\bar{Z},T)$
$-a^r(\nabla K)(X,\bar{Y}(\nabla F)(Z,S),T) - a^r(\nabla K)(X,\bar{Y}(\nabla F)(Z,T),S) - a^r(K)(X,(\nabla F)(Y,S),(\nabla F)(Z,T)$
$-a^r(K)(X,(\nabla F)(Y,T),(\nabla F)(Z,S) + K((\nabla F)(\bar{X},T),\bar{Y},(\nabla F)(Z,S))$
$+K((\nabla F)(\bar{X},S),\bar{Y},(\nabla F)(Z,T)) + K((\nabla F)(\bar{X},T),(\nabla F)(Y,S),\bar{Z})$
$+K((\nabla F)(\bar{X},S),(\nabla F)(Y,T),\bar{Z}) + K((\nabla\nabla F)(\bar{X},T,S),\bar{Y},\bar{Z})$
$-a^r K((X,(\nabla\nabla F)(Y,T,S),\bar{Z}) - a^r K((X,\bar{Y}(\nabla\nabla F)(Z,T,S)$
$-(\nabla V)((\nabla F)(\bar{X},S),\bar{Y},\bar{Z},T) + a^r(\nabla V)(X,(\nabla F)(Y,T),\bar{Z},S)$
$+a^r(\nabla V)(X,(\nabla F)(Y,S),\bar{Z},T) + a^r(\nabla V)(X,\bar{Y}(\nabla F)(Z,S),T)$
$+a^r(\nabla V)(X,\bar{Y}(\nabla F)(Z,T),S) + a^r(V)(X,(\nabla F)(Y,S),(\nabla F)(Z,T))$
$+a^r(V)(X,(\nabla F)(Y,T),(\nabla F)(Z,T)) - V((\nabla F)(\bar{X},T),\bar{Y}(\nabla F)(Z,S))$
$-V((\nabla F)(\bar{X},S),\bar{Y}(\nabla F)(Z,T)) - V((\nabla F)(\bar{X},T),(\nabla F)(Y,S),\bar{Z})$

$-V((\nabla F)(\bar{X},S),(\nabla F)(Y,T),\bar{Z}) - V((\nabla\nabla F)(\bar{X},T,S),\bar{Y},\bar{Z})$
$+a^r V(X,(\nabla\nabla F)(Y,T,S),Z)+a^r V(X,\bar{Y}(\nabla\nabla F)(Z,T,S)))\}$ (2.20)
Adding equation (2.19) from equation (2.20),we get

$$
\begin{aligned}
&-a^r(\nabla\nabla C)(X,\bar{Y},\bar{Z},T,S)+(\nabla C)\big((\nabla F)(\bar{X},T),\bar{Y},\bar{Z},S\big)+(\nabla C)(\nabla F)(\bar{X},S),\bar{Y},\bar{Z},T)\\
&-a^r(\nabla C)(X,(\nabla F(Y,T),\bar{Z},S)-a^r(\nabla C)(X,(\nabla F(Y,S),\bar{Z},T)-a^r(\nabla C)(X,\bar{Y}(\nabla F(Z,S),T)\\
&-a^r(\nabla C)(X,\bar{Y}(\nabla F(Z,T),S)-a^r C(X,(\nabla F(Y,S),(\nabla F)(Z,T)\\
&-a^r C(X,(\nabla F(Y,T),(\nabla F)(Z,S)+C((\nabla F(\bar{X},T),\bar{Y},(\nabla F)(Z,S)\\
&+C((\nabla F(\bar{X},S),\bar{Y},(\nabla F)(Z,T)+C((\nabla F(\bar{X},T),(\nabla F)(Y,S),\bar{Z})\\
&+C\big((\nabla F(\bar{X},S),(\nabla F)(Y,T),\bar{Z}\big)+C\big((\nabla F)(\bar{X},T),(\nabla F)(Y,S),Z\big)\\
&+C\big((\nabla F)(\bar{X},S),(\nabla F)(Y,T),Z\big)+C((\nabla\nabla F(\bar{X},T,S),\bar{Y},Z)-a^r C(X,\nabla\nabla F)(Y,T,S),Z\\
&+C((\nabla\nabla F(\bar{X},T,S),\bar{Y},\bar{Z})-a^r C(X,\nabla\nabla F)(Y,T,S),\bar{Z})-a^r C(X,\bar{Y}(\nabla\nabla F)(Z,T,S)\\
&+a^r P_2(T,S)L(X,\bar{Y},\bar{Z})\\
&=\\
&-a^r(\nabla\nabla L)(X,\bar{Y},\bar{Z},T,S)+(\nabla L)\big((\nabla F)(\bar{X},T),\bar{Y},\bar{Z},S\big)+(\nabla L)\big((\nabla F)(\bar{X},S),\bar{Y},\bar{Z},T\big)\\
&-a^r(\nabla L)(X,(\nabla F)(Y,T),\bar{Z},S)-a^r(\nabla L)(X,(\nabla F)(Y,S),\bar{Z},T)-a^r(\nabla L)(X,\bar{Y}(\nabla F)(Z,S),T)\\
&-a^r(\nabla L)(X,\bar{Y}(\nabla F)(Z,T),S)-a^r L(X,(\nabla F)(Y,S),(\nabla F)(Z,T)-a^r L(X,(\nabla F)(Y,T),(\nabla F)(Z,S)\\
&+L((\nabla F)(\bar{X},T),\bar{Y},(\nabla F)(Z,S))+L((\nabla F)(\bar{X},S),\bar{Y},(\nabla F)(Z,T))+L((\nabla F)(\bar{X},T),(\nabla F)(Y,S),\bar{Z})\\
&+L((\nabla F)(\bar{X},S),(\nabla F)(Y,T),\bar{Z})+L((\nabla\nabla F)(\bar{X},T,S),(\bar{Y},,\bar{Z})-a^r L(X,(\nabla\nabla F)(Y,T,S),\bar{Z})\\
&-a^r L(X,\bar{Y}(\nabla\nabla F)(Z,T,S)+\frac{n}{(n-2)}\{-a^r(\nabla\nabla K)(X,\bar{Y},\bar{Z},T,S)+(\nabla K)(\nabla F)(\bar{X},T),\bar{Y},\bar{Z},S)\\
&+(\nabla K)(\nabla F)(\bar{X},S),\bar{Y},\bar{Z},T)-a^r(\nabla K)(X,(\nabla F)(Y,T),Z,S)-a^r(\nabla K)(X,(\nabla F)(Y,S),\bar{Z},T)\\
&-a^r(\nabla K)(X,\bar{Y}(\nabla F)(Z,S),T)-a^r(\nabla K)(X,\bar{Y}(\nabla F)(Z,T),S)-a^r(K)(X,(\nabla F)(Y,S),(\nabla F)(Z,T)\\
&-a^r(K)(X,(\nabla F)(Y,T),(\nabla F)(Z,S)+K\big((\nabla F)(\bar{X},T),\bar{Y},(\nabla F)(Z,S)\big)\\
&+K\big((\nabla F)(\bar{X},S),\bar{Y},(\nabla F)(Z,T)\big)+K\big((\nabla F)(\bar{X},T),(\nabla F)(Y,S),\bar{Z}\big)\\
&+K\big((\nabla F)(\bar{X},S),(\nabla F)(Y,T),\bar{Z}\big)+K((\nabla\nabla F)(\bar{X},T,S),\bar{Y},\bar{Z})\\
&-a^r K((X,(\nabla\nabla F)(Y,T,S),\bar{Z})-a^r K((X,\bar{Y}(\nabla\nabla F)(Z,T,S)+a^r P_2(T,S)K(X,\bar{Y},\bar{Z})\\
&+a^r(\nabla\nabla V)(X,\bar{Y},\bar{Z},T,S)-(\nabla V)((\nabla F)(\bar{X},T),\bar{Y},\bar{Z},S)-(\nabla V)((\nabla F)(\bar{X},S),\bar{Y},\bar{Z},T)\\
&+a^r(\nabla V)(X,(\nabla F)(Y,T),\bar{Z},S)+a^r(\nabla V)(X,(\nabla F)(Y,S),\bar{Z},T)+a^r(\nabla V)(X,\bar{Y}(\nabla F)(Z,S),T)\\
&+a^r(\nabla V)(X,\bar{Y}(\nabla F)(Z,T),S)+a^r(V)(X,(\nabla F)(Y,S),(\nabla F)(Z,T))\\
&+a^r(V)(X,(\nabla F)(Y,T),(\nabla F)(Z,T))-V((\nabla F)(\bar{X},T),\bar{Y}(\nabla F)(Z,S))\\
&-V((\nabla F)(\bar{X},S),\bar{Y}(\nabla F)(Z,T))-V((\nabla F)(\bar{X},T),(\nabla F)(Y,S),\bar{Z})\\
&-V((\nabla F)(\bar{X},S),(\nabla F)(Y,T),\bar{Z})-V((\nabla\nabla F)(\bar{X},T,S),\bar{Y},\bar{Z})\\
&+a^r V(X,(\nabla\nabla F)(Y,T,S),Z)+a^r V(X,\bar{Y}(\nabla\nabla F)(Z,T,S))\}-a^r P_2(T,S)V(X,\bar{Y},\bar{Z})\}.
\end{aligned}
$$

If a (1,2,3) –birecurrent H-HSU –structure manifold is conformal-(1,2,3)-birecurrent and conharmonic-(1,2,3)-birecurrent then from equation (2.21),we have

$$
\begin{aligned}
&+a^r(\nabla\nabla V)(X,\bar{Y},\bar{Z},T,S)-(\nabla V)((\nabla F)(\bar{X},T),\bar{Y},\bar{Z},S)-(\nabla V)((\nabla F)(\bar{X},S),\bar{Y},\bar{Z},T)\\
&+a^r(\nabla V)(X,(\nabla F)(Y,T),\bar{Z},S)+a^r(\nabla V)(X,(\nabla F)(Y,S),\bar{Z},T)+a^r(\nabla V)(X,\bar{Y}(\nabla F)(Z,S),T)\\
&+a^r(\nabla V)(X,\bar{Y}(\nabla F)(Z,T),S)+a^r(V)(X,(\nabla F)(Y,S),(\nabla F)(Z,T))\\
&+a^r(V)(X,(\nabla F)(Y,T),(\nabla F)(Z,T))-V((\nabla F)(\bar{X},T),\bar{Y}(\nabla F)(Z,S))\\
&-V((\nabla F)(\bar{X},S),\bar{Y}(\nabla F)(Z,T))-V((\nabla F)(\bar{X},T),(\nabla F)(Y,S),\bar{Z})\\
&-V((\nabla F)(\bar{X},S),(\nabla F)(Y,T),\bar{Z})-V((\nabla\nabla F)(\bar{X},T,S),\bar{Y},\bar{Z})\\
&+a^r V(X,(\nabla\nabla F)(Y,T,S),Z)+a^r V(X,\bar{Y}(\nabla\nabla F)(Z,T,S))=a^r P_2(T,S)V(X,\bar{Y},\bar{Z})\}.
\end{aligned}
$$

Which shows,that the manifolds is concicular-(1,2,3)- birecurrent.
Similarly ,it can be shown that if the manifold is either conformal-(1,2,3)-birecurrent and concircular –(1,2,3) –birecurrent then it is either conharmonic –(1,2,3)- birecurrent and concircular-(1,2,3)-birecurrent then it is either conharmonic-(1,2,3)-birecrrent then it is either conharmonic-(1,2,3)-birecurrent or conformal-(1,2,3)-birecurrent for the same recurrence parameter.

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