

“ON CHARACTERISTIC PROPERTIES OF α -COMPACTNESS IN FUZZY TOPOLOGICAL SPACES”

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Abstract- Our main aim is to show Fuzzy set theory of various mathematical structures, whose features emphasize the effects of ordered structure, may be developed on the Fuzzy topological theory. Fuzzy topology is one such branch, combining ordered with topological structure. This is wider than that of classical set theory. We also introduce the concept of fuzzy nearly C- α -compactness in fuzzy topological spaces and fuzzy bitopological spaces for deriving some interesting applications of fuzzy almost continuous and fuzzy almost open functions. We also derive some basic characterisation of C- α -compactness in fuzzy topological space by C. L. Chang ⁽¹⁾. C. L. Chang and Zadeh ⁽³⁾ developed the concept of fuzzy topological spaces based on the concept of a fuzzy set.

Key-words: Fuzzy Topological space, compactness, compact space, α -Open, Set theory and regular closed set.

1 INTRODUCTION:

We introduce the concept of fuzzy nearly C- α -compactness in fuzzy bitopological spaces. We state some properties related with the concept of fuzzy nearly C- α -compactness and some of the characterizations in fuzzy topological spaces of fuzzy almost α -continuous, fuzzy almost α -open functions. A fuzzy topological space (X, T) is said to be fuzzy compact if every cover of X by members of T has a finite subcover. Also, x_α denote a fuzzy point with a support x and value α ($0 < \alpha \leq 1$). For a fuzzy set λ in X , Let us write $x_\alpha \in \lambda$ provided $\alpha = \lambda(x)$. A topological space (X, T) is said to be nearly C- α -compact if for any ordinary subset A of X , $A \neq X$ such that χ_A (the characteristic function of $A \subset X$) is a proper regular closed set and for each open cover of $\{\lambda_\alpha; \alpha \in \Delta\}$ of χ_A there exists a finite subfamily $\{\lambda_{\alpha_1}, \lambda_{\alpha_2}, \dots, \lambda_{\alpha_n}\}$ such that $\chi_A \subset \bigcup_{i=1}^n Cl(\lambda_{\alpha_i})$. The concept of nearly C-compactness in general topology has studied by P. L. Sharma. Let x denote fuzzy topological space, whereas by a fuzzy set in a non-void set X stand for a function λ from X to the unit closed interval $[0, 1]$ ($= I$) i.e., $\lambda \in I^X$. By x_t . Fuzzy set whose value at the singleton support $\{x\}$ is t and 0 for a fuzzy set λ in X , $1-\lambda$ denotes the fuzzy complement of λ and is defined by $(1-\lambda)(x) = 1 - \lambda(x)$, for each $x \in X$. The notations $Cl(\lambda)$ and $Int(\lambda)$ will stand for the closure and interior respectively of the fuzzy set λ in the fts X . For any family $\{\lambda_i; i \in \Lambda\}$ of fuzzy sets in X , the union $\bigvee_{i \in \Lambda} \lambda_i$ and $\bigwedge_{i \in \Lambda} \lambda_i$ are defined as $(\bigvee_{i \in \Lambda} \lambda_i)(x) = \sup_{i \in \Lambda} \lambda_i(x)$ and $(\bigwedge_{i \in \Lambda} \lambda_i)(x) = \inf_{i \in \Lambda} \lambda_i(x)$ for each $x \in X$, where Λ is an arbitrary index set A of X is denoted by χ_A and is defined as

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \quad \dots(1.1)$$

Let $U = \{\lambda_\alpha\}_{\alpha \in \Delta}$ be a family of members from T . Then U is called a cover of X if $\bigvee_{\alpha \in \Delta} \lambda_\alpha = 1$ and a subfamily of U having a similar property is called a subcover of U .

2 C- α -Compactness in Fuzzy Topological Spaces:

These definition shows that fuzzy compactness implies fuzzy nearly C- α -compactness. A topological space is said to be nearly C- α -compact if given a regular closed set A and an α -open cover $U = \{\lambda_i; i \in \Lambda\}$ of A there exists a finite subfamily $\{\lambda_i; i = 1, 2, \dots, n\}$ of U such that $A \subset \bigcup_{i=1}^n Cl_\alpha(\lambda_i)$. Let (X, T) be a fuzzy topological space. (X, T) is said to be fuzzy nearly C- α -compact if for any ordinary subset A of X , $A \neq X$ such that χ_A (the characteristic function of $A \subset X$) is a proper fuzzy regular closed set and for each fuzzy α -open cover of $\{\lambda_\alpha; \alpha \in \Delta\}$ of χ_A there exists a finite subfamily $\{\lambda_{\alpha_1}, \lambda_{\alpha_2}, \dots, \lambda_{\alpha_n}\}$ such that $\chi_A \leq \bigvee_{i=1}^n Cl_\alpha(\lambda_{\alpha_i})$.

Let $X = \{a, b\}$, $T = \{0, 1, f_n\}$ where $f_n: X = \{a, b\} \rightarrow [0, 1]$ is such that $f_n(X) = 1 - \frac{1}{n}$, $\forall x \in X$. The only possible non-empty subsets of X are $A_1 = \{a\}$ and $A_2 = \{b\}$. Further, since $Cl Int \chi_{A_1} = 0 \neq \chi_{A_1}$ and $Cl Int \chi_{A_2} = 0 \neq \chi_{A_2}$, it follows that χ_{A_1} and χ_{A_2} are not fuzzy regular closed. So vacuously (X, T) is fuzzy nearly C- α -

compact. Now we claim that (X, T) is not fuzzy compact. Indeed, of $\bigvee_{n=1}^{\infty} f_n = 1$ shows that $\{f_n\}_{n=1}^{\infty}$ is a fuzzy open cover of 1_X but for every infinite integer, say n_0 , we have $\bigvee_{n=1}^{n_0} f_n < 1$ and therefore $\{f_n\}_{n=1}^{\infty}$ has no finite subcover for 1_X . That is (X, T) is not fuzzy compact.

3 Properties of Fuzzy Almost α -Continuous and α -Open :

Let (X, T) and (Y, S) be any two fuzzy topological spaces. A mapping $f : (X, T) \rightarrow (Y, S)$ is said to be fuzzy almost α -open (α -closed) if the image of every fuzzy regular open (closed) set is fuzzy α -open (α -closed). Let (X, T) and (Y, S) be any two fuzzy topological spaces. A mapping $f : (X, T) \rightarrow (Y, S)$ is said to be fuzzy almost α -continuous if the inverse image of every fuzzy regular open (closed) set is fuzzy α -open (α -closed). Let $X = \{a, b, c\}$;

Let us define $T_1 = \{0, 1, \lambda\}$ and $T_2 = \{0, 1, \mu\}$ where $\lambda(a) = 0, \lambda(b) = \frac{2}{3}, \lambda(c) = \frac{1}{2}$ and $\mu(a) = 1, \mu(b) = 0, \mu(c) = 0$. Let $f : (X, T_1) \rightarrow (X, T_2)$ be the identity mapping. In (X, T_2) the only non-zero fuzzy regular open set is 1 and $f^{-1}(1) = 1$ shows that f is fuzzy almost α -continuous. Now let $g : (X, T_2) \rightarrow (X, T_1)$ be the identity mapping, the only non-zero fuzzy regular open set in (X, T_2) is 1 and $g(1) = 1$ implies that f is fuzzy almost α -open.

4 Theorem:

The image of a fuzzy nearly C- α -compact space under a fuzzy almost α -continuous and fuzzy almost α -open mapping is fuzzy nearly C- α -compact.

Proof

Let $f : (X, T) \rightarrow (Y, S)$ be a fuzzy almost α -continuous and fuzzy almost α -open mapping from a fuzzy nearly C- α -compact X onto Y . It is to be proved that Y is also fuzzy nearly C- α -compact. Let A be any subset of Y such that χ_A is fuzzy regular closed in Y . Let $U = \{\lambda_i\}_{i \in \Delta}$ be a fuzzy regular α -open cover of χ_A in Y . $f^{-1}(\chi_A)$ is a fuzzy regular closed subset of X and $\{f^{-1}(\lambda_i)\}_{i \in \Delta}$ is a fuzzy regular α -open cover of $f^{-1}(\chi)$ in X . Since X is fuzzy nearly C- α -compact, there exists a finite subfamily $\{f^{-1}(\lambda_i); i = 1, 2, \dots, n\}$ such that

$$f^{-1}(\chi_A) \leq \bigvee_{i=1}^n cl_{\alpha}\{f^{-1}(\lambda_i)\} \leq \bigvee_{i=1}^n \{f^{-1}(cl_{\alpha}(\lambda_i))\} \quad \dots (1.2)$$

That is $\chi_A = \bigvee_{i=1}^n \{cl_{\alpha}(\lambda_i)\}$. This proves that Y is fuzzy nearly C- α -compact.

5 Comparative structure of C- α -Compactness in Fuzzy Bitopological Spaces:

The concept of fuzzy bitopological spaces was introduced by A. Kandel and subsequently further studied. By A. E. Sheikn. A fuzzy bitopological space is an ordered triple (X, T_1, T_2) where T_1 and T_2 are fuzzy topologies on X . A bitopological space (X, T_1, T_2) is said to be (1, 2)-fuzzy nearly C-compact if for every set $A \subset X$ such that χ_A is a proper T_1 fuzzy regular closed set and for every T_2 -fuzzy open cover U of χ_A , there exists a finite sub collection of U , (say) $\lambda_1, \lambda_2, \dots, \lambda_n$ such that $\chi_A \leq \bigvee_{i=1}^n T_2 - cl(\lambda_i)$. Then (X, T_1, T_2) is said to be pairwise fuzzy nearly C-compact if it is both (1, 2)-fuzzy nearly C-compact and (2, 1)-fuzzy nearly C-compact. A bitopological space (X, T_1, T_2) is said to be (1, 2)-fuzzy nearly C- α -compact if for every set $A \subset X$ such that χ_A is a proper T_1 -fuzzy regular closed set and for every T_2 -fuzzy α -open cover U of χ , there exists a finite sub-collection of U , (say) $\lambda_1, \lambda_2, \dots, \lambda_n$ such that $\chi_A =$. Then (X, T_1, T_2) is said to be pairwise fuzzy nearly C- α -compact if it is both (1, 2)-fuzzy nearly C- α -compact and (2, 1)-fuzzy nearly C- α -compact.

6 Theorem:

Every pairwise fuzzy α -continuous and pairwise fuzzy almost α -open image of a pairwise fuzzy nearly C- α -compact space is pairwise fuzzy nearly C- α -compact.

Proof

Let $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$ be any pairwise fuzzy almost α -continuous and pairwise fuzzy almost α -open onto mapping. Assume (X, T_1, T_2) is pairwise fuzzy nearly C- α -compact. We show that (Y, S_1, S_2) is pairwise fuzzy nearly C- α -compact.

Let $A \subset Y$ be such that χ_A is a proper S_1 -fuzzy regular closed set and let U be a S_2 -fuzzy α -open cover of χ_A . Since f is fuzzy almost α -continuous and fuzzy almost α -open, $f^{-1}(\chi_A)$ is T_1 -fuzzy regular closed and $\{f^{-1}(\mu) : \mu \in U\}$ is a T_2 - α -open cover of $f^{-1}(\chi_A)$. Since (X, T_1, T_2) is pairwise fuzzy nearly C- α -compact, there exists a finite sub-collection $\{f^{-1}(\mu_k) : k = 1, 2, \dots, n\}$ such that $f^{-1}(\chi_A) \leq \bigvee_{k=1}^n T_2 - cl_{\alpha} f^{-1}(\mu_k)$. Hence, we have

$$\chi_A = f f^{-1}(\chi_A) \leq \bigvee_{k=1}^n f [T_2 - cl_{\alpha} f^{-1}(\mu_k)]$$

$$\leq \bigvee_{k=1}^n S_2 - cl_{\alpha}(ff^{-1}(\mu_k)) \leq \bigvee_{k=1}^n S_2 - cl_{\alpha}(\mu_k) \quad \dots (1.3)$$

This proves that Y is $(1, 2)$ -fuzzy nearly C - α -compact. We may show that Y is also $(2, 1)$ -fuzzy nearly C - α -compact. Thus it has been shown that (Y, S_1, S_2) is pairwise fuzzy nearly C - α -compact. Hence, the theorem is proved.

7 Theorem:

Let (X, T_1, T_2) be any pairwise fuzzy nearly C - α -compact space. Then if $A \subset X$ is such that χ_A is proper T_1 -fuzzy regular closed and \mathfrak{F} is a family of T_j -fuzzy α -closed subsets of X such that $\bigwedge \{\lambda \wedge \chi_A; \lambda \in \mathfrak{F}\} = 0$, there exists a finite number of elements, say $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of \mathfrak{F} such that

$$\bigwedge_{k=1}^n \{T_j - Int_{\alpha} \lambda_k \wedge \chi_A\} = 0, i \neq j, i, j = 1, 2 \quad \dots (1.4)$$

Proof

We suppose (X, T_1, T_2) is to be fuzzy pairwise nearly C - α -compact. Let $A \subset X$ be such that χ_A is proper T_1 -fuzzy regular closed and \mathfrak{F} is a family of T_j -fuzzy α -closed sets of X such that $\bigwedge \{\lambda \wedge \chi_A; \lambda \in \mathfrak{F}\} = 0$. Now $\bigwedge_{\lambda \in \mathfrak{F}} \{\lambda \wedge \chi_A\} = 0 \Rightarrow \bigwedge_{\lambda \in \mathfrak{F}} \lambda \leq \chi_A \Rightarrow \chi_A \leq 1 - \bigwedge_{\lambda \in \mathfrak{F}} \lambda = \bigvee_{\lambda \in \mathfrak{F}} (1 - \lambda) = \bigvee_{\lambda \in \mathfrak{F}} \{1 - \lambda; \lambda \in \mathfrak{F}\}$. So $\{1 - \lambda; \lambda \in \mathfrak{F}\}$ is a T_j -fuzzy α -open cover of χ_A which is T_1 -fuzzy regular closed and hence by assumption we have a finite collection, say $1 - \lambda_1, 1 - \lambda_2, \dots, 1 - \lambda_n$ such that $\chi_A \leq \bigvee_{k=1}^n T_j - cl_{\alpha}(1 - \lambda_k) = \bigvee_{k=1}^n (1 - T_j - Int_{\alpha} \lambda_k) = 1 - \bigwedge_{k=1}^n T_j - Int_{\alpha} \lambda_k$. This implies $\bigwedge_{k=1}^n T_j - Int_{\alpha} \lambda_k \leq 1 - \chi_A = \chi_{X-A}$. Therefore, $\chi_A \wedge (\bigwedge_{k=1}^n T_j - Int_{\alpha} \lambda_k) \leq \chi_A \wedge \chi_{X-A} = 0 \Rightarrow \bigwedge_{k=1}^n (T_j - Int_{\alpha} \lambda_k \wedge \chi_A) = 0$. Hence, the theorem is established.

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