Some Approximate Fixed-Point Theorems in metric spaces

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Abstract- **Many problems in pure and applied mathematics have as their solutions the fixed point of some mapping F. Therefore a number of procedures in numerical analysis and approximations theory amount to obtaining successive approximations to the fixed point of an approximate mapping. Our object in this paper to discuss about fixed point theory and its applications in metric spaces, also we established some fixed-point theorems in complete metric spaces, which generalized many results of great mathematician.**

Keyword- **Fixed-point theory, Metric-space, b-Metric space, 2-metric space, Approximate fixed point, contraction mapping and operators.**

INTRODUCTION-

Let T be a self map of a metric space (X, d). Let us look for an approximate solution of Tx = x. If there exists a point z ∈ X such that d(Tz, z) ≤ ε , where ε is a positive number, then z is called an approximate solution of the equation , Tx = x or equivalently, z ∈ X is an approximate fixed point (or ε -fixed point) of T . In many situations of practical utility, the mapping under consideration may not have an exact fixed point due to some tight restriction on the space or the map, or an approximate fixed point is more than enough, an approximate solution plays an important role in such situations. The theory of fixed points and consequently of approximate fixed points finds application in mathematical economics, non cooperative game theory, dynamic programming, nonlinear analysis, variational calculus, theory of differential equations and several other areas of applicable analysis (see, for instance, [5], [6], [8], [10], [11] and several references thereof).

Cromme and Diener [7] have found approximate fixed points by generalizing Brouwer's fixed point theorem to a discontinuous map, Hou and Chen have extended their results to set valued maps. Espinola and Kirk obtained interesting results in product spaces. Tijs et al [12]have discussed approximate fixed point theorems for contractive and non-expansive maps by weakening the conditions on the spaces. R. Branzei et al [5] further extended these results to multifunctions in Banach spaces. Recently M. Berinde [3] obtained approximate fixed point theorems for operators satisfying Kannan, ChatterjeaandZamfirescu type of conditions on metric spaces. In this paper we study some basic approximate fixed point results in generalized metric spaces.

PRELIMINARIES-

<u>FIXED POINT</u>:-

Let T: <u>X</u>→ X be a function defined on set X. A point of X which remains invariant under the transformation T, is called a fixed point that is T (x) = x, fixed point of a function can also be defined as a point that is mapped onto itself.

Example:

1. Translation has no fixed point. F(x) = x + c; c is non zero.
2. $F(x) = x^2$; 0 and 1 are fixed points.
3. $F(x) = x^3$; 0, -1 and 1 are fixed points.
4. $F(x) = x^2+x-1$, then 1 is a fixed point of f, because F(x) =1.
5. $F(x) = (x^2+8)/6$, then 2 is a fixed point of f, because F(x) =2.

<u>METRIC SPACE</u>:-

Let X be a non –empty set then a mapping d: X × X →R is said to be metric on X, where X is any set, if for any x , y, z ∈ X , following conditions hold;

(M1) d(x, y) ≥0 i.e.d is real valued, finite and non-negative.
(M2) d(x, y) = 0 if and only if x = y
(M3) d(x, y) = d(y, x) (symmetry)
(M4) d(x, y) ≤ d (x, z) + d (z, y) (Triangle Inequality)

Then the pair (X, d) is called a metric space.

<u>APPROXIMATE FIXED POINT:-</u>

Let T: X →X, $\in > 0$ and $x_0 \in X$ .Then an element $x_0 \in X$ is an approximate fixed point (or ϵ-fixed point) of T if$d(Tx_0, x_0) < \in$.

<u>APPROXIMATE FIXED POINT PROPERTY:-</u>

Let T: X →X, Then T has the approximate fixed point property (a. f .p. p.) if$\forall \in > 0, F_{\in}(f) \neq \Phi$

CONTRACTION MAPPING:-

Let X =(X, d) be a metric space .A mapping T: X→X is called a contraction on X .if there is a positive real number $\alpha < 1$ such that for all x, y Є X.

d $(T_x, T_y) \leq \alpha$d(x, y)

Contraction mapping are continuous but converse is not true.

WEAK CONTRACTION:-

A function T of a metric space (X, d) into itself is said to be a weak contraction if for all x, y ∈ X.d (T(x), T(y)) ≤ d(x, y)
Obviously every contraction is a weak contraction .the converse is false.

COMMUTING MAPPING:-

Let (X, d) be a metric space and A, S be two self maps defined on X, then A and S is said to be commuting if

AS =SA

WEAKLY COMMUTING MAPPING:-

Any two self maps A and S of (X, d) are said to be weakly commuting if
d(ASx, SAx) ≤ d(Ax, Sx) for all x in X.

Example:-

Let X =[0 ,1] define A and S : X → X by
A(x) = $\frac{x}{2-x}$ and S(x) =$\frac{x}{2}$ for all x in X.
Then are weakly commuting but not commuting.

Solution:-

We show that
d (ASx ,SAx) ≤ d(Ax ,Sx)
but AS ≠ SA

Now A(x) = $\frac{x}{2-x}$, S(x) =$\frac{x}{2}$

We have

$$AS(x) = \frac{x/2}{2-x/2} = \frac{x}{4-x}$$

And

$$SA(x) = \frac{x\backslash 2-x}{2} = \frac{x}{4-2x}$$

Then

d(ASx , SAx) =$\left|\frac{x}{4-x} - \frac{x}{4-2x}\right| = \frac{x^2}{(4-x)(4-2x)}$

and

d(Ax, Sx) =$\left|\frac{x}{2-x} - \frac{x}{2}\right| = \frac{x^2}{(4-2x)}$

=> $\frac{x^2}{(4-x)(4-2x)} \leq \frac{x^2}{(4-2x)}$

So,

d(ASx , SAx) ≤ d(Ax ,Sx)

and

$$\frac{x}{4-x} \neq \frac{x}{4-2x}$$

Clearly

AS ≠ SA

Here A and S are weakly commuting but not commuting.

2-metric space:-

A 2-metric space is a set X with a function d:X x X x X→[0, ∞) satisfying the Following condition:

(1) For two distinct point x, y ∈ X ,there exists a point z ∈ X such that d(x, y ,z)≠ 0
(2) d(x, y, z)=0 if at least two of x, y, z are equal .
(3) d(x, y, z)=d(x, z, y)=d(y, z, x)
(4) d(x, y, z) ≤ d(x, y, u) +d(x, u ,z) +d(u, y , z) for all x, y, z, u ∈ X

The function d is called 2-metric for the space X and (X, d) denoted
2- Metric space.

a-CONTRACTIONS:-

A mapping f: X → X is an a-contraction if ∃a ∈]0, 1[such that
d(f(x),f(y)) ≤ ad(x, y) ,∀ x, y ∈ X.

KANNAN OPERATOR:-

A mapping f: X → X is a Kannan operator if ∃a ∈]0, $\frac{1}{2}$ [such that
d (f(x), f(y)) ≤ a [d(x, f(x)) + d(y, f(y))], ∀ x, y ∈ X.

CHATTERJEA OPERATOR:-

A mapping f:X→ X is a Chatterjea operator if ∃ a ∈]0, $\frac{1}{2}$ [such that
d (f(x), f(y)) ≤ a [d(x, f(y) + d(y, f(x))], ∀ x, y ∈ X.

Bakhtin [4] introduced the concept of b-metric space as a generalization of metric space.

b-metric spaces:-

Let s ≥ 1 be a given real numbers. A function d: X x X → R_+ (set of non Negative real numbers) is said to be a b-metric if for all x , y, z, ∈ X The Following conditions are satisfied. Let X be a non empty set

(1) d(x, y) = 0 if x = y,
(2) d(x, y) = d(y, x),
(3) d(x, z) ≤ s [d(x, y) + d(y, z)] ,

A Pair (X, d) is called a b-metric space.

ASYMPTOTICALLY REGULAR:-

A mapping T: X→X is said to be asymptotically regular if for any
d(T^nx,T^{n+1}x) → 0 as n→ ∞∀ x ∈ X .
Then T has approximate fixed point property.

Theorem1:

Let (X, d) be a metric space, f: X→X such that f is asymptotic regular, i.e.,
d (f^n(x), f^{n+1}(x)) → 0 as n→ ∞ ,∀ x ∈ X .
Then f has the approximate fixed point property.

Proof. Let x_0 ∈ X.
Then
d ($f^n(x_0)$, $f^{n+1}(x_0)$) → 0 as n→ ∞
∀ ∈>0, ∃$n_0(\epsilon)$ ∈ N^* such that ∀ n ≥ $n_0(\epsilon)$, d ($f^n(x_0)$,$f^{n+1}(x_0)$) < ∈
∀ ∈>0, ∃$n_0(\epsilon)$ ∈ N^* such that ∀ n ≥ $n_0(\epsilon)$, d ($f^n(x_0)$,$f(f^n(x_0)$) < ∈
Denoting

$y_0 = f^n(x_0)$,
It follows that
∀ϵ >0, ∃ y_o ∈ X such that d (y_0,$f(y_0)$) <∈

So for each ∈ > 0 there exists an ∈-fixed point of f in X, namely y_0
This means exactly that f has the approximate fixed point property.

Theorem2:

Let (X, d) be a b-metric space and T: X →X if T is a kannan operator. then∀ ∈> 0, $F_\epsilon(f)$≠ Φ and d(x ,y) ≤ $s \in (1+3sa)$

Proof:

Let x ∈ X and ∈ > 0,
Then we have

$$d(f^n(x),f^{n+1}(x)) = d(f(f^{n-1}(x)),f(f^n(x)))$$
$$\le a\,[d(f^{n-1}(x), f(f^{n-1}(x))) + d(f^n(x), f(f^n(x)))]$$
$$= ad(f^{n-1}(x), f^n(x)) + ad(f^n(x), f^{n+1}(x)$$
$$(1-a)d\,(f^n(x), f^{n+1}(x)) \le ad(f^{n-1}(x), f^n(x))$$
$$d(f^n(x), f^{n+1}(x)) \le \frac{a}{1-a}\, d(f^{n-1}(x), f^n(x))$$
$$\le \left(\frac{\mathbf{a}}{\mathbf{1-a}}\right)^2 \boldsymbol{d(f^{n-2}(x), f^{n-1}(x))}$$

:
:
:

Continuing this for n time we finally see.

$$d(f^n(x), f^{n+1}(x)) \le \left(\frac{a}{1-a}\right)^n d(x, f(x))$$

But a ∈ (0, 1/2)
$d(f^n(x), f^{n+1}(x)) \to 0$, as n→ ∞ ,$\forall x \in X$

BY using lemma 3.1 it follow that $F_{\in}(f) \neq \Phi$, $\forall \in > 0$.

Let x and y are two $\in -$ fixed point of f.

Then

$$d(x,y) \leq d(x,f(x)) + d(f(x),y)$$
$$\leq s[d(x,f(x)) + d(f(x),y)]$$
$$s \geq 1$$
$$\leq s\in + sd(f(x),y)]$$
$$\leq s\in + s[s(d(f(x),f(y) + d(f(y),y)]$$
$$\leq s\in + s^2\, d(f(x),f(y)) + s^2\, d(f(y),y)$$

By definition of kannan operator

$$\leq s\in + s^2[ad(x,f(x)) + ad(y,f(y))] + s^2\, d(f(y),y))$$
$$\leq s\in +s^2 a\in +s^2 a\in +s^2 a\in$$
$$\leq s\in +3s^2 a\in$$
$$d(x,y) \leq s\in(1+3sa)$$

Theorem3

Let (X, d) be a b-metric space and T: X →X if T is a chatterjea operator.

Then:

$\forall \in > 0, F_{\in}(f) \neq \Phi$ and $d(x,y) \leq \frac{s\in(1+3as)}{1-2as^2}$

Proof:

Let x ∈ X and ∈ > 0,

Then we have

$$d(f^n(x),f^{n+1}(x)) = d(f(f^{n-1}(x)),f(f^n(x)))$$
$$\leq a[d(f^{n-1}(x),f(f^{n-1}(x)) + d(f^n(x),f(f^n(x))]$$
$$= ad(f^{n-1}(x),f^n(x)) + ad(f^n(x),f^{n+1}(x))$$
$$(1-a)d(f^n(x),f^{n+1}(x)) \leq ad(f^{n-1}(x),f^n(x))x^2$$
$$d(f^n(x),f^{n+1}(x)) \leq \frac{a}{1-a}\, d(f^{n-1}(x),f^n(x))$$
$$\leq \left(\frac{\mathbf{a}}{\mathbf{1-a}}\right)^2 \boldsymbol{d(f^{n-2}(x),f^{n-1}(x))}$$

:
:
:

Continuing this for n time we finally see.

$$d(f^n(x),f^{n+1}(x)) \leq \left(\frac{a}{1-a}\right)^n d(x,f(x))$$

But $a \in (0,1/2)$

$d(f^n(x),f^{n+1}(x)) \to 0$, as n→ ∞ ,$\forall x \in X$

BY using lemma 3.1 it follow that $F_{\in}(f) \neq \Phi$, $\forall \in > 0$.

Let x and y are two $\in$ fixed point of f.

Then

$$d(x,y) \leq d(x,f(x)) + d(f(x),y)$$
$$\leq s[d(x,f(x)) + d(f(x),y)]$$
$$s \geq 1$$
$$\leq s\in + sd(f(x),y)]$$
$$\leq s\in + s[s(d(f(x),f(y) + d(f(y),y)]$$
$$\leq s\in + s^2\, d(f(x),f(y)) + s^2\, d(f(y),y)$$

By definition of chatterjea operator

$$\leq s\in +s^2[ad(x,f(y)) + ad(y,f(x))] + s^2 d(f(y),y)$$
$$\leq s\in +s^2\, ad(x,f(y)) + s^2\, ad(y,f(x)) + s^2\, d(f(y),y)$$
$$\leq s\in +as^2[d(x,y) + d(y,f(y))] + as^2[d(y,f(y)) + d(f(y),f(x))] + s^2\, d(f(y),y)$$

Using by Triangle inequality (metric property)

$$\leq s\in +as^2\, d(x,y) + as^2\, d(y,f(y)) + as^2\, d(y,f(y)) + as^2\, d(f(x),f(y)) + s^2\, d(f(y),y)$$
$$\leq s\in +as^2\, d(x,y) + as^2 \in +as^2 \in +as^2\, d(f(x),f(y)) + as^2 \in$$
$$\leq s\in +as^2\, d(x,y) + 3as^2 \in +as^2\, d(f(x),f(y))$$

By definition of weak contraction

$$d(f(x),f(y)) \leq d(x,y)$$
$$\leq s\in +as^2\, d(x,y) + 3as^2 \in +as^2\, d(x,y)$$

$$d(x,y) \leq \; s \in +3as^2 \; \in +2as^2\, d(x,y)$$
$$(1-2as^2)d(x,y) \leq \; s \in (1+3as)$$
$$d(x,y) \leq \frac{s \in (1+3as)}{1-2as^2}$$

This completes the proof.

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