# A Study on Univalent Functions and their Geometrical Properties in Geometric Function theory

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*Abstract-* **"Geometric function theory is the branch of complex analysis which deals with the study of geometrical properties of analytic functions. It is an area of mathematics characterized by fascinating nuptials between geometry and analysis. The theory of univalent and multivalent analytic functions of a complex variable come under the scope of geometric function theory. Complex Analysis is traditionally known as the theory of function of a complex variable. It is the branch of mathematical analysis that investigates functions of complex numbers. It is useful in many branches of mathematics including algebraic geometry, number theory, analytic combinatory, applied mathematics as well as in physics, including hydrodynamics and thermodynamics and also in engineering field such as nuclear, aerospace, mechanical and electrical engineering by extension. The main aim of this paper is to study Analytic functions, Univalent functions, Analytic-Univalent functions. To investigate some geometrical properties of Analytic-Univalent functions. To study Starlike function and convex function."**

*Key words*: **Geometric function theory, Analytic function, Univalent function, Koebe function, Starlike function, Convex function.**

## Introduction:

**"**Geometric function theory is the branch of complex analysis which deals with the study of geometric properties of analytic functions**.** It is an area of mathematics characterized by fascinating nuptials between geometry and analysis. The theory of univalent and multivalent analytic functions of a complex variable come under the scope of geometric function theory. **C**omplex Analysis is traditionally known as the theory of function of a complex variable. It is the branch of mathematical analysis that investigates functions of complex numbers. It is useful in many branches of mathematics including algebraic geometry, number theory, analytic combinatory, applied mathematics as well as in physics, including hydrodynamics and thermodynamics and also in engineering field such as nuclear, aerospace, mechanical and electrical engineering by extension. Complex analysis is particularly concerned with analytic functions of complex variables. Because the real and imaginary parts of analytic function must satisfy Laplace's equation. Complex analysis is widely applicable to two-dimensional problem in physics. Complex analysis is a branch of mathematics that investigates the analytical properties of complex variable functions .It is an elegant and powerful method useful in the study of heat flow, fluid dynamics and electrostatics. Two dimensional potential problems can be solved using analytic function since the real and imaginary part of analytic function are the solutions of the two dimensional Laplace's equation".

## Cauchy-Riemann Equations:

"Let $f(z) = u(x, y) + iv(x, y)$ be a complex valued function and is differentiable at a point $z_0 = x_0 + iy_0$ ,then both $u(x, y)$ and $v(x ,y)$ have first order partial derivatives at $z_0 = x_0 + i y_0$ .

and $u_x(x_0,y_0) = v_y(x_0,y_0)$ and $u_y(x_0,y_0) = - v_x(x_0,y_0)$ This is called Cauchy –Riemann equations".

## Analytic Function:

"A single valued function $w= f(z)$ which is defined and differentiable at each point of the domain D is called an analytic function in that domain D.

A function $f(z)$ is said to be an analytic function at $z_0$ ,if it is differentiable at each point in some neighborhood of $z_0$ . Analytic function is also called holomorphic function or regular function or monogenic function. ".

**Necessary condition for the function to be Analytic function**:

"If $w= f(z) = u + iv$ is an analytic function in the domain D,

Then i) The first order partial derivatives of $u(x, y)$ and $v(x, y)$ must exist.

ii) It must satisfy the Cauchy Riemann Equations

i. e $u_x = v_y$ and $u_y = -v_x$."

**Sufficient condition for the function to be Analytic function:**

"A single valued function $w= f(z)$ is said to be analytic in the domain D,

If i) The four partial derivatives $u_x$, $u_y$, $v_x$ ,$v_y$ exist and continuous.

ii) The four partial derivatives $u_x$, $u_y$, $v_x$ ,$v_y$ must satisfy Cauchy-Riemann equations at every point of the domain , i.e $u_x = v_y$ and $u_y = - v_x$. ".

**Examples:**

i) All polynomial functions.
ii) The exponential functions
iii) The trigonometric function, logarithmic function, Exponential functions are analytic in their domain.

**Riemann Mapping Theorem**: “It states that there exists a unique function f which maps the domain $D \subset C$ conformally on to the unit disk $\Delta = \{z \in C : |Z| < 1\}$ and
$f(z) = 0$ , $f'(z) > 0$ for a given point $z \in \Delta$ “ .

**Univalent Function:**
“A single valued function f(z) is said to be an univalent function in a domain D ,
If $f(z_1) = f(z_2) \Rightarrow z_1 = z_2$ for all points $z_1, z_2 \in D$ . In other words f (z) is said to be univalent in D, If it does not assume any complex value more than once in D. The function is said to be locally univalent at a point $z_0 \in D,$ if it is univalent in some neighborhood of $z_0$.
The necessary condition for an analytic function $f(z)$ is said to be univalent in D,
if $f'(z) \neq 0 \; \forall \; z \in D$ .
The sufficient condition is not true. So, Every analytic function need not be univalent and every univalent function need not be analytic.
The function $f(z) = e^z$ is not univalent in C and its derivative never vanishes in C.
The function $h(z) = \frac{1}{z}$ is univalent function in any domain containing the point $z = 0$
but not analytic at $z = 0$ .
The only functions that are analytic and univalent in the complex plane are of the form $(z) = az + b$ , Where a, b are complex constants and $a \neq 0$.”

**Normalized condition of analytic-univalent function:**
The normalized condition of analytic and univalent function is $(0) = 0 \; and \; f'(0) = 1$ .
The class of analytic function is denoted by A and the sub-class of analytic function which is univalent is denoted by S in the unit disk $\Delta$ $(|z| \leq 1)$.
The function f(z) is called univalent in $\Delta$ ,if $f'(z) \neq 0$. The univalent function with normalized condition $f(0) = 0 \; and \; f'(0) = 1$ is given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ .

**Example:** Koebe function is the leading example of the class S i.e univalent function . It is denoted by K(z). $K(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right] = \sum_{n=1}^{\infty} nz^n$ , $|z| < 1$ .
Koebe function maps the disk D on to the entire plane minus the part of the negative real axis from $-\frac{1}{4} \; to \; \infty$ .
Kobe function is the largest univalent function .It plays a central role as an extremal function for many subclasses of univalent function.
The subclasses of Analytic function related to Univalent function are Starlike function and Convex function.

**Starlike function**: “A function $f \in S$ is said to be starlike with respect to origin ,if it maps from $\Delta$ to a starlike domain with respect to origin i.e the line segment joining w=0 to any point of $f(\Delta)$ lies in $f(\Delta)$ . The class of univalent starlike function is denoted by $S^*$ ".

**Necessary and Sufficient condition for a univalent function to be Starlike function:**
“A necessary and sufficient condition for a function $f \in S$ to be an univalent starlike function in $\Delta$ , if $Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \;\; (z \in \Delta)$.
Geometrically for each $r \; (0 < r < 1)$ , $\arg\left(f\left(re^{i\theta}\right)\right)$ increases with $\theta \;\; (0 \leq \theta \leq 2\pi)$".

**Convex function** : “A function $f \in S$ is said to be convex function in $\Delta$ , if it maps $\Delta$ on to a convex domain. i.e The straight line segment joining any two points of $f(\Delta)$ is contained in $f(\Delta)$ . The class of univalent convex function is denoted by K" .

**Necessary and Sufficient condition for a univalent function to be convex function:**
“A necessary and sufficient condition for a function $f \in S$ to be an univalent convex function in $\Delta$ , if $Re\left\{1 + \frac{zf'(z)}{f(z)}\right\} > 0 \;\; (z \in \Delta)$.
Geometrically the function $w = f(z),\; z = re^{i\theta}, z \in \Delta$ is convex in $\Delta$ if and only if the slope of the tangent to the curve increases $w = f(re^{i\theta})$ , $(0 \leq \theta \leq 2\pi)$ is an increasing function of $\theta$ .
Every convex function is a starlike function i.e K $\subset S^*$ .
Koebe function is a univalent starlike function .so, K$(z) \in S^*$ ”.
A function $f \in S$ is said to be univalent convex of order , $(0 \leq \alpha \leq 1$ ,
if $Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \;\; (z \in \Delta)$. The class of univalent Starlike function of order $\alpha$ is denoted by $S^*(\alpha)$.
Every convex function is Starlike of order $\frac{1}{2}$.
A function $f \in S$ is said to be univalent Starlike of order $\alpha$ , $(0 \leq \alpha \leq 1$ ,
if $Re\left\{1 + \frac{zf'(z)}{f(z)}\right\} > \alpha \;\; (z \in \Delta)$. The class of univalent convex function of order $\alpha$ is denoted by K$(\alpha)$.

Close to Convex function : A function $f \in A$ is said to be close to convex function, if there exists a function g $\in S^*$ and a complex number $\varepsilon$ , $|\varepsilon| = 1$ such that

$Re\left\{\varepsilon \frac{zf'(z)}{g(z)}\right\} > 0 \ (z \in \Delta)$. The class of close to convex function is denoted by $\mathcal{C}$.

Close to Convex function is a subset of Starlike function.$i.e$ $\mathcal{C} \subset S^*$ ".

**Conclusion:**

"The study of analytic function can be used in heat conduction, fluid flows and in different fields of physics and engineering. By using Analytic function we can connect different branches of mathematics .The idea of analytic function includes the whole wealth of functions, most important to science. A complex function which is analytic at all points is called an entire function. Complex analysis is an important part of the mathematical landscape as it connects many topics. It can be used as important topic for independent research. The complex method is a general optimization technique that can be used to solve a wide range of nonlinear problems directly. The study of univalent function ,Starlike function ,convex function can be used in different fields of Mathematics. Physics and Engineering."

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