On Pseudo Slant Submanifolds of Nearly Quasi Sasakian Manifolds with Quarter Symmetric Metric Connection

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Abstract: **The object of the present paper is to study pseudo slant submanifolds of nearly quasi Sasakian manifolds admitting quarter symmetric metric connection. We study necessary and sufficient conditions on totally umbilical proper-slant submanifolds and obtain some results on such submanifolds. We also discuss the integrability conditions of distributions of pseudo-slant submanifolds of nearly quasi Sasakian manifolds with quarter symmetric metric connection.**

***Key words*:** **Nearly quasi-Sasakian manifold, slant submanifold, proper slant submanifold, pseudo slant submanifold, quarter symmetric metric connection.**

1. Introduction

The notion of a slant submanifold as a natural generalization of both holomorphic and totally real immersion was given by Chen [8]. Many authors have studied slant immersion in almost Hermitian manifold. A Lotta [16] introduced the notion of slant immersion in contact manifold. The properties of slant submanifold of an almost contact manifolds were studied by Lotta [16]. L. Cabrerizo et al. [10] was defined slant submanifold of Sasakian manifolds. N. Papaghiuc [17] introduced and studied the notion of semi slant submanifold of an almost Hermitian manifold. A Carrizo [12, 13, 14] defined Hemi slant submanifolds. The contact version of Pseudo slant submanifolds in a Sasakian manifolds have been studied by V. A. Khan et.al. [15] and the author studied nearly quasi Sasakian manifold.

In Section 2, we recall some results and formula for later use. In Section 3, we define a pseudo-slant submanifold of a nearly quasi-Sasakian manifold and in Section 4, it is concern with the integrability of the distributions on pseudo-slant submanifolds of a nearly quasi-Sasakian manifold with quarter symmetric metric connection and obtains some characterizations. In Section 5, we prove the classification theorem for totally umbilical pseudo-slant submanifolds of a nearly quasi-Sasakian manifold with quarter symmetric metric connection.

2. Preliminaries.

Let $\bar{M}$ be a real 2n+1 dimensional differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g). then we have

$$\phi^2 = -I + \eta \otimes \xi \ , \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \phi\xi = 0 \quad \eta o \phi = 0$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\phi X, Y) = -g(X, \phi Y) \tag{2.1}$$

For any vector field X, Y tangent to $\bar{M}$, where I is the identity on the tangent bundle $\Gamma\bar{M}$ of $\bar{M}$. An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called quasi Sasakian manifold if

$$(\bar{\nabla}_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \qquad \phi AX = A\phi X \tag{2.2}$$

Where A is symmetric linear transformation field is $\bar{\nabla}$ denotes the Riemannian connection of g on $\bar{M}$.
Further, an almost contact metric manifold $\bar{M}$ on (ϕ, ξ, η, g) is called nearly quasi-Sasakian manifold if

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = \eta(Y)AX + \eta(X)AY - 2g(AX, Y)\xi \tag{2.3}$$

We have also on a quasi-Sasakian manifold $\bar{M}$

$$\bar{\nabla}_X \xi = \phi AX \tag{2.4}$$

A quarter symmetric metric connections is defined as

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi \tag{2.5}$$

The covariant derivative of the tensor field ϕ is defined as

$$(\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y \tag{2.6}$$

Using equation (2.1), (2.2), and (2.5) in equation (2.6), we get

$$(\bar{\nabla}_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi - g(X, Y)\xi + \eta(Y)X \tag{2.7}$$

Thus in particular, an almost contact metric manifold $\bar{M}$ on (ϕ, ξ, η, g) is called nearly quasi- Sasakian manifold with quarter symmetric metric connection if,

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = \eta(Y)(AX + X) + \eta(X)(AY - Y)$$
$$-2g(AX, Y)\xi - 2g(X, Y)\xi \tag{2.8}$$

Now, let M be a submanifold immersed in $\bar{M}$. The Riemannian metric induced on M is denoted by the same symbol g. Let PM and $P^{\perp}M$ be the Lie algebras of vectors fields tangential to M and normal to M respectively and ∇ be the induced Levi-Civita connection on M, then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.9}$$

$$\bar{\nabla}_X V = -A_V X + \nabla_X^{\perp} V + \eta(V)\phi X \tag{2.10}$$

For any $X, Y \in PM$ and $V \in P^{\perp}M$, where $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M, h$ is the second fundamental form and A_V is the Weingarten map associated with V as

$$g(A_V X, Y) = g(h(X, Y), V) \tag{2.11}$$

For any $X \in PM$ and $V \in P^{\perp}M$, we write

$$\phi X = PX + VX, \quad (PX \in PM \ and \ VX \in P^{\perp}M) \tag{2.12}$$

$$\phi V = tV + nV, \quad (tV \in PM \ and \ nV \in P^{\perp}M) \tag{2.13}$$

The submanifold M is invariant if N is identically zero. On the other hand, M is anti- invariant if T is identically zero. From (2.1) and (2.12), we have

$$g(X, PY) = -g(PX, Y). \tag{2.14}$$

For any $X, Y \in PM$. if we put $Q = P^2$, we have

$$(\nabla_X Q)Y = \nabla_X QY - Q\nabla_X Y \tag{2.15}$$

$$(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y \tag{2.16}$$

$$(\nabla_X V)Y = \nabla_X^{\perp} VY - V\nabla_X Y \tag{2.17}$$

For any $X, Y \in PM$. In view of (2.9), (2.12) and (2.4) it follows that

$$\bar{\nabla}_X \xi = PAX, \tag{2.18}$$

$$h(X, \xi) = VAX \tag{2.19}$$

The mean curvature vector H of M is given by

$$H = \frac{1}{n} trace(h) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \tag{2.20}$$

Where n is the dimension of M and $e_1, e_2, e_3 \dots \dots \dots, e_n$ is a local coordinate frame of M. A submanifold of a contact manifold $\bar{M}$ is said be totally umbilical if

$$h(X, Y) = g(X, Y)H \tag{2.21}$$

A submanifold M is said to be totally geodesic if $h(X, Y) = 0$ for any $X, Y \in \Gamma(PM)$ and M is said to be minimal if $H = 0$.

3. Pseudo-Slant Submanifolds of Nearly Quasi-Sasakian Manifolds.

The purpose of this section is to study the existence of pseudo-slant submanifolds of nearly quasi-Sasakian manifolds.

A Lotta [16] introduced the notion of slant immersion and the properties of slant submanifold in almost contact metric manifolds. L. Cabrerizo et al. [10] was defined slant submanifold of Sasakian manifolds.

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be a slant submanifold if for any $x \in M$ and $X \epsilon P_x(M)$, linearly independent on ξ, the angle between ϕX and $P_x(M)$ is a constant. The constant angle $\theta(x) \in [0, \frac{\pi}{2}]$ is called slant angle of M in $\bar{M}$.

A submanifold M of nearly quasi-Sasakian manifold $\bar{M}$ is said to be pseudo-slant submanifold if there exists two orthogonal distributions $D^{\perp}$ and D_θ on M such that

(i). TM has the orthogonal direct decomposition, i.e. $PM = D^{\perp} \oplus D_\theta \oplus \langle \xi \rangle,\ \xi \epsilon \Gamma(D_\theta)$.

(ii).The distribution $D^{\perp}$ is an anti-invariant. i.e. $\phi D^{\perp} \subset P^{\perp}M$

(iii).The distribution D_θ is a slant, i.e. the slant angle between D_θ and $\phi(D_\theta)$ is a constant.

From above definition, it is clear that if $\theta = 0$, then the pseudo slant submanifold is a semi invariant submanifolds and if $\theta = \frac{\pi}{2}$, and then submanifold becomes an anti-invariant.

On the other hand, we suppose that M is a pseudo slant submanifold of nearly quasi Sasakian manifold $\bar{M}$ and we denote the dimension of distribution $D^{\perp}$ and D_θ by d_1 and d_2 respectively, then we have the following cases:

1). If $d_2 = 0$, then M is an anti-invariant submanifold.

2). If $d_1 = 0$ and $\theta = 0$, then M is an invariant submanifold.

3). If $d_1 = 0$ and $\theta \neq 0$, then M is a proper slant submanifold with slant angle θ.

4). If $d_1 . d_2 \neq 0$ and $\theta \in \left[0, \frac{\pi}{2}\right]$, then M is a proper pseudo- slant submanifold.

Theorem: (3.1) Let M be a submanifold of a nearly quasi Sasakian manifold $\bar{M}$ such that $\xi \in PM$, then M is slant iff there exists a constant $\lambda \epsilon [0, 1]$ such that

$$P^2 = -\lambda\{I - \eta \otimes \xi\} \tag{3.1}$$

Furthermore, in such a case if θ is the slant angle of M, then $\lambda = cos^2\theta$.

Corollary 3.2 Let M be a slant submanifold of a nearly quasi–Sasakian manifold $\bar{M}$ with slant angle θ, then for any $X, Y \in \Gamma(PM)$, we have

$$g(PX, PY) = cos^2\theta(g(X, Y) - \eta(X)\eta(Y)) \tag{3.2}$$

$$g(VX, VY) = sin^2\theta(g(X, Y) - \eta(X)\eta(Y)) \tag{3.3}$$

Let M be a proper slant slant submanifold of a contact manifold $\bar{M}$ and the projection on $D^{\perp}$ and D_θ By P_1 and P_2 respectively, then for any vector field $X \epsilon \Gamma(PM)$, we can write

$$X = P_1X + P_2X + \eta(X)\xi \tag{3.4}$$

Now applying ϕ both sides of (3.4), we obtain

$$\phi X = \phi P_1X + \phi P_2X$$

That is,

$$PX + VX = V P_1X + P P_2X + VP_2X \tag{3.5}$$

We can easily to see

$$PX = P P_2X, VX = V P_1X + VP_2X \tag{3.6}$$

$$\phi P_1X = V P_1X,\ T P_1X = 0,\quad \phi P_2X = TP_2X + VP_2X \tag{3.7}$$

$$TP_2X \in \Gamma(D_\theta) \tag{3.8}$$

If we denote the orthogonal complementary of ϕPM in $D^{\perp}M$ by μ, then the normal bundle $P^{\perp}M$ can be decomposed as follows

$$P^{\perp}M = V(D^{\perp}) \oplus V(D_\theta) \oplus \mu \tag{3.9}$$

Where μ is the invariant sub bundle of $P^{\perp}M$ as $V(D_\theta)$ are orthogonal distribution on M. indeed, $g(Z, X) = 0$, For each $Z \in \Gamma(D^{\perp})$ and $X \in \Gamma(D_\theta)$, thus by equation (2.1) and (2.12), we can write

$$g(VZ, VX) = g(\phi Z, \phi X) = g(Z, X) = 0 \tag{3.10}$$

That is the distribution $V(D^{\perp})$ and $V(D_\theta)$ are mutually perpendicular. In fact, the decomposition (3.9) is an orthogonal direct decomposition.

4. Integrability of the Distributions of Pseudo-Slant Submanifolds of Nearly Quasi-Sasakian Manifolds with Quarter Symmetric Metric Connection.

In this section we will discuss the integrability conditions of the distributions of pseudo-slant submanifolds of nearly quasi-Sasakian manifolds with quarter symmetric metric connection.

Theorem 4.1 Let M be a pseudo-slant submanifold of nearly quasi–Sasakian manifold $\bar{M}$ with quarter symmetric metric connection. Then for all $X, Y \epsilon D^{\perp}$ we have

$$A_{\phi Y}X - A_{\phi X}Y = g(AX,Y)\xi - \nabla_X(PY) - h(X,PY) + A_{VY}X - \nabla^{\perp}_{X}(VX)$$
$$+P(\nabla_X Y) + V(\nabla_X Y) + V\big(h(X,Y)\big). \quad (4.1)$$

Proof: In view of (2.11), we get

$$g\big(A_{\phi Y}X,Z\big) = g(h(X,Z),\phi Y) = -g(\phi h(X,Z),Y) \quad (4.2)$$

From (2.9) and (4.2), we get

$$g\big(A_{\phi Y}X,Z\big) = -g(\phi\overline{\nabla}_Z X,Y) + g(\phi\nabla_Z X,Y)$$
$$= -g(\phi\overline{\nabla}_Z X,Y)$$
$$= g((\overline{\nabla}_Z\phi)X,Y) - g(\overline{\nabla}_Z\,\phi X,Y). \quad (4.3)$$

Now for $X \in D^{\perp}$, $\phi X \in P^{\perp}M$. Hence from (2.10) we have

$$\overline{\nabla}_Z\phi X = -A_{\phi X}Z + \nabla^{\perp}_{Z}\phi X + \eta(Z)\phi^2 X$$
$$= -A_{\phi X}Z + \nabla^{\perp}_{Z}\phi X - \eta(Z)X + \eta(X)\eta(Z)\xi \quad (4.4)$$

Combining (4.3) and (4.4) we obtain

$$g\big(A_{\phi Y}X,Z\big) = g\big((\overline{\nabla}_Z\phi)X,Y\big) + g\big(A_{\phi X}Z,Y\big)$$
$$-\eta(Z)g(X,Y) + \eta(X)\eta(Z)\eta(Y) \quad (4.5)$$

Since $h(X,Y) = h(Y,X)$ it follows from (2.11)

$$g\big(A_{\phi X}Z,Y\big) = g\big(A_{\phi X}Y,Z\big).$$

Hence from (4.5) we obtain with the help of (2.8)

$$g\big(A_{\phi Y}X,Z\big) - g\big(A_{\phi X}Y,Z\big) = g((\overline{\nabla}_Z\phi)X,Y) - \eta(Z)g(X,Y) - \eta(X)\eta(Y)\eta(Z)$$
$$(\overline{\nabla}_Z\phi)X + (\overline{\nabla}_X\phi)Z = \eta(Z)AX + \eta(X)\,AZ - 2g(AX,Z)\xi - 2g(X,Z)\xi$$
$$-\eta(X)Z + \eta(Z)X$$
$$= \eta(Z)g(AX,Y) + \eta(X)g(AZ,Y) - 2g(AX,Z)\eta(Y)$$
$$-2g(X,Z)\eta(Y) - \eta(X)g(Z,Y) + \eta(Z)g(X,Y) - g((\overline{\nabla}_X\phi)Z,Y)$$
$$-\eta(Z)g(X,Y) + \eta(X)\eta(Y)\eta(Z)$$

Therefore above equation become

$$g\big(A_{\phi Y}X,Z\big) - g\big(A_{\phi X}Y,Z\big) = \eta(Z)g(AX,Y) + g(\nabla_X(PY) + h(X,PY)$$
$$-A_{VY}X + \nabla^{\perp}_{X}VX - \eta(X)\phi VY - P(\nabla_X Y) - V(\nabla_X Y)$$
$$-P(h(X,Y) - V(h(X,Y))),Z) \quad (4.6)$$

Since $X,Y,Z \in D^{\perp}$ an orthogonal distribution to the distribution $\{\xi\}$, it follows that $\eta(X) = \eta(Y) = 0$. Therefore above equation (4.6) become

$$A_{\phi Y}X - A_{\phi X}Y = g(AX,Y)\xi - \nabla_X(PY) - h(X,PY)$$
$$+A_{VY}X - \nabla^{\perp}_{X}(VX) + P(\nabla_X Y) + V(\nabla_X Y) + V\big(h(X,Y)\big)$$

Theorem 4.2 In a pseudo–slant submanifold of nearly quasi–Sasakian manifold $\overline{M}$ with quarter symmetric metric connection is given by

$$(\nabla_X P)Y = A_{VY}X + A_{VX}Y + th(X,Y) + P\big(h(Y,X)\big) - (\nabla_Y P)X + \eta(Y)AX$$
$$+\eta(X)AY - 2g(AX,Y)\xi - 2g(X,Y)\xi - \eta(X)Y + \eta(Y)X \quad (4.7)$$

Proof. Let $X,Y \in PM$, we have

$$\overline{\nabla}_X\phi Y = (\overline{\nabla}_X\phi)Y + \phi(\overline{\nabla}_X Y) \text{ and } \overline{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

From (2.12) and (2.13), we obtain

$$\overline{\nabla}_X PY + \overline{\nabla}_X VY = (\overline{\nabla}_X\phi)Y + \phi\nabla_X Y + \phi h(X,Y)$$

Also from (2.12) and (2.13), we obtain

$$\overline{\nabla}_X PY + \overline{\nabla}_X VY = (\overline{\nabla}_X\phi)Y + P(\nabla_X Y) + V(\nabla_X Y) + th(X,Y) + nh(X,Y).$$

Using (2.9), (2.10) and (2.8), from above we get

$$\nabla_X PY + h(X,PY) - A_{VY}X + \nabla^{\perp}_{X}(VY) + \eta(X)\phi VY = \eta(Y)AX + \eta(X)AY$$
$$- 2g(AX,Y)\xi - 2g(X,Y)\xi - \eta(X)Y + \eta(Y)X - \overline{\nabla}_Y\phi X + \phi(\overline{\nabla}_Y X)$$
$$+ P(\nabla_X Y) + V(\nabla_X Y) + th(X,Y) + nh(X,Y) - \nabla_Y PX - h(Y,PX)$$
$$+A_{VX}Y - \nabla^{\perp}_{Y}VX + \eta(Y)\phi VX + P(\nabla_Y X) + V\nabla_Y X$$
$$+P(h(Y,X) + V(h(Y,X)) \quad (4.8)$$

Comparing tangential and normal parts we get

$$\nabla_X PY - A_{VY}X = \eta(Y)AX + \eta(X)AY - 2g(AX,Y)\xi$$
$$-2g(X,Y)\xi - \eta(X)Y + \eta(Y)X$$

$$-\nabla_Y PX + A_{VX}Y + P(\nabla_Y X) + P(h(Y,X)) + P(\nabla_X Y) + th(X,Y) \tag{4.9}$$

That is,

$$(\nabla_X P)Y = A_{VY}X + A_{VX}Y + th(X,Y) + P\big(h(Y,X)\big) - (\nabla_Y P)X + \eta(Y)AX$$
$$+\eta(X)AY - 2g(AX,Y)\xi - 2g(X,Y)\xi - \eta(X)Y + \eta(Y)X \tag{4.10}$$

Theorem 4.3 Let M be a pseudo- slant submanifold of nearly quasi –Sasakian manifold $\bar{M}$ with quarter symmetric metric connection. Then the anti-invariant distribution $D^{\perp}$ is integrable if and only if for any $Z, W \in \Gamma(D^{\perp})$.

$$A_{VW}Z + A_{VZ}W + 2T\nabla_Z W + 2th(W,Z) = -\eta(W)AZ - \eta(Z)AW$$
$$+2g(AZ,W)\xi + 2g(Z,W)\xi + \eta(Z)W - \eta(W)Z \tag{4.11}$$

Proof: Let $Z, W \epsilon \Gamma(D^{\perp})$ and using (2.8), we obtain

$$(\bar{\nabla}_Z\phi)W + (\bar{\nabla}_W\phi)Z = \eta(W)AZ + \eta(Z)AW - 2g(AZ,W)\xi$$
$$-2g(Z,W)\xi - \eta(Z)W + \eta(W)Z$$

Which is equivalent to

$$\bar{\nabla}_Z\phi W - \phi\bar{\nabla}_Z W + \bar{\nabla}_W\phi W - \phi\bar{\nabla}_W Z = \eta(W)AZ + \eta(Z)AW - 2g(AZ,W)\xi$$
$$-2g(Z,W)\xi - \eta(Z)W + \eta(W)Z$$

Using (2.9), (2.10), (2.12) and (2.13), we obtain

$$\eta(W)AZ + \eta(Z)AW - 2g(AZ,W)\xi - 2g(Z,W)\xi - \eta(Z)W + \eta(W)Z$$
$$= \bar{\nabla}_Z NW - T\nabla_Z W - V\nabla_Z W - th(W,Z)$$
$$-nh(W,Z) + \bar{\nabla}_W NZ - T\nabla_W Z - V\nabla_W Z - th(W,Z) - nh(W,Z)$$

So we have,

$$\eta(W)AZ + \eta(Z)AW - 2g(AZ,W)\xi - 2g(Z,W)\xi - \eta(Z)W + \eta(W)Z$$
$$= -A_{VW}Z + \nabla^{\perp}_Z(VW) - \eta(Z)\phi VW - T\nabla_Z W - V\nabla_Z W - 2th(W,Z)$$
$$-A_{VZ}W + \nabla^{\perp}_W(VZ) - \eta(W)\phi VZ - T\nabla_W Z - V\nabla_W Z - 2nh(W,Z)$$

Now comparing tangential and normal parts we get,

$$A_{VW}Z + A_{VZ}W + T\nabla_Z W + T\nabla_W Z + 2th(W,Z)$$
$$= -\eta(W)AZ - \eta(Z)AW + 2g(AZ,W)\xi + 2g(Z,W)\xi + \eta(Z)W - \eta(W)Z$$

From above we can infer

$$-\eta(W)AZ - \eta(Z)AW + 2g(AZ,W)\xi + 2g(Z,W)\xi + \eta(Z)W - \eta(W)Z$$
$$= A_{VW}Z + A_{VZ}W + 2T\nabla_Z W - T(\nabla_Z W - \nabla_W Z) + 2th(W,Z)$$
$$T[Z,W] = A_{VW}Z + A_{VZ}W + 2T\nabla_Z W + 2th(W,Z)$$
$$+\eta(W)AZ + \eta(Z)AW - 2g(AZ,W)\xi - 2g(Z,W)\xi - \eta(Z)W + \eta(W)Z$$

Thus $[Z,W] \in \Gamma(D^{\perp})$ if and only if (4.11) is satisfied.

Theorem 4.4 Let M be a Pseudo-slant submanifold of a nearly quasi–Sasakian manifold $\bar{M}$ with quarter symmetric metric connection. Then the slant distribution D_θ is integrable if and only if for any $X, Y \in \Gamma(D_\theta)$

$$P_1\{\nabla_X(PY) - P\nabla_Y X + (\nabla_Y P)X - A_{VX}Y - A_{VY}X - 2th(X,Y)$$
$$-\eta(Y)AX - \eta(X)AY + \eta(X)Y - \eta(Y)X\} = 0 \tag{4.12}$$

Proof: For any $X, Y \in \Gamma(D_\theta)$ we denote the projections on $D^{\perp}$ and D_θ by P_1 and P_2 Respectively, Then for any vector fields $X, Y \in \Gamma(D_\theta)$, by using (2.8) we get

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = \eta(Y)AX + \eta(X)AY - 2g(AX,Y)\xi$$
$$-2g(X,Y)\xi - \eta(X)Y + \eta(Y)X$$
$$\bar{\nabla}_X\phi Y - \phi\bar{\nabla}_X Y + \bar{\nabla}_Y\phi X - \phi\bar{\nabla}_Y X = \eta(Y)AX + \eta(X)AY - 2g(AX,Y)\xi$$
$$-2g(X,Y)\xi - \eta(X)Y + \eta(Y)X$$

Using (2.9), (2.10), (2.12) and (2.13), we obtain

$$\bar{\nabla}_X PY + \bar{\nabla}_X VY - \phi(\nabla_X Y + h(X,Y) + \bar{\nabla}_Y PX + \bar{\nabla}_Y VX - \phi(\nabla_Y X + h(X,Y)$$
$$= \eta(Y)AX + \eta(X)AY - 2g(AX,Y)\xi - 2g(X,Y)\xi$$
$$-\eta(X)Y + \eta(Y)X$$
$$\nabla_X PY + h(X,PY) - A_{VY}X + \nabla^{\perp}_X(VY) - \eta(X)\phi VX - P\nabla_X Y - V\nabla_X Y - th(X,Y)$$
$$- nh(X,Y) + \nabla_Y PX + h(Y,PX) - A_{VX}Y + \nabla^{\perp}_Y(VX)$$
$$-\eta(X)\phi VX - P\nabla_Y X - V\nabla_Y X - th(X,Y) - nh(X,Y)$$
$$= \eta(Y)AX + \eta(X)AY - 2g(AX,Y)\xi - 2g(X,Y)\xi - \eta(X)Y + \eta(Y)X \tag{4.13}$$

From tangent component of (4.4.13), we get

$$\nabla_X PY - P\nabla_X Y + (\nabla_Y P)X - A_{VX}Y - A_{VY}X - 2th(X,Y) \tag{4.14}$$

$$= \eta(Y)AX + \eta(X)AY - \eta(X)Y + \eta(Y)X$$

$$P[X,Y] = \nabla_X PY - P\nabla_X Y + (\nabla_Y P)X - A_{VX}Y - A_{VY}X - 2th(X,Y) \quad (4.15)$$
$$-\eta(Y)AX - \eta(X)AY + \eta(X)Y - \eta(Y)X$$

Applying P_1 to (4.15), we get (4.12).

Theorem 4.5 Let M be a Pseudo- slant submanifold of a nearly quasi Sasakian manifold $\bar{M}$ with quarter symmetric metric connection, Then the distribution $D^{\perp} \oplus \ <\xi>$ is integrable if and only if for any $Z, W \ \in \Gamma\,(D^{\perp} \oplus \ <\xi>)$,

$$[A_{\phi Z}W - A_{\phi W}Z] = \frac{1}{3}\begin{bmatrix}\eta(AZ)W - \eta(Z)AW + 2\eta(Z)W + \eta(W)AZ - \eta(AW)Z \\ -\eta(AW)Z - 2\eta(W)Z\end{bmatrix}$$

Proof: For any $Z, W \ \in \Gamma\,(D^{\perp} \oplus \ <\xi>)$ and $U \ \in \ \Gamma\,(PM)$, by using (2.11) we can write

$$2g\big(A_{\phi Z}W, U\big) = g(h(U,W),\phi Z) + g(h(U,W),\phi Z).$$

By using (2.9), we have

$$2g\big(A_{\phi Z}W, U\big) = g(\bar{\nabla}_W U, \phi Z) + g(\bar{\nabla}_U W, \phi Z)$$
$$= -g(\phi\bar{\nabla}_W, Z) - g(\phi\bar{\nabla}_U W, Z)$$
$$2g\big(A_{\phi Z}W, U\big) = g((\bar{\nabla}_W\phi)U + (\bar{\nabla}_U\phi)W, Z) - g(\bar{\nabla}_W\phi U, Z) - g(\bar{\nabla}_U\phi W, Z)$$

By using (2.8), we obtain

$$2g\big(A_{\phi Z}W, U\big) = -g(\bar{\nabla}_W\phi U, Z) - g(\bar{\nabla}_U\phi W, Z)$$
$$+g(\eta(U)AW + \eta(W)AU - 2g(AW,U)\xi, Z) - 2g(W,U) - \eta(W)U + \eta(U)W, Z)$$
$$= -g(\bar{\nabla}_W Z, \phi U) - g\big(-A_{\phi W}U, Z\big) + g(\eta(AW)Z, U) + g(\eta(W)AZ, U)$$
$$-2g(\eta(AW)Z, U) - 2g(\eta(W)Z, U) - g(\eta(W)Z, U) + g(\eta(W)Z, U)$$
$$= -g(P\nabla_W Z + th(\mathrm{Z},\mathrm{W}), \mathrm{U}) + g\big(A_{\phi W}Z, U\big) + g(\eta(AW)Z, U)$$
$$+g(\eta(W)AZ, U) - 2g(\eta(AW)Z, U) - 2g(\eta(W)Z, U) - g(\eta(W)Z, U)$$
$$+g(\eta(W)Z, U)$$
$$2A_{\phi Z}W = -P\nabla_W Z - th(\mathrm{Z},\mathrm{W}) + A_{\phi W}Z + \eta(AW)Z + \eta(W)AZ - 2\eta(AW)Z$$
$$-2\eta(W)Z - \eta(W)Z + \eta(W)Z$$

This is equivalent to,

$$2A_{\phi Z}W = \eta(W)AZ - \eta(AW)Z + A_{\phi W}Z - 2\eta(W)Z$$
$$- P\nabla_W Z - th(\mathrm{Z},\mathrm{W}) \quad (4.16)$$

Take $Z = W$

$$2A_{\phi W}Z = \eta(Z)AW - \eta(AZ)W + A_{\phi Z}W - 2\eta(Z)W$$
$$-P\nabla_Z W - th(\mathrm{W},\mathrm{Z}) \quad (4.17)$$

By using (4.16) and (4.17), we obtain

$$3\big(A_{\phi Z}W - A_{\phi W}Z\big) = P\nabla_Z W - P\nabla_W Z + th(\mathrm{W},\mathrm{Z}) - th(\mathrm{Z},\mathrm{W}) - \eta(Z)AW$$
$$+\eta(AZ)W + 2\eta(Z)W + \eta(W)AZ - \eta(AW)Z - 2\eta(W)Z$$
$$= P[Z,W] - \eta(Z)AW + \eta(AZ)W + 2\eta(Z)W + \eta(W)AZ$$
$$-\eta(AW)Z - 2\eta(W)Z$$

$$\big(A_{\phi Z}W - A_{\phi W}Z\big) = \frac{1}{3}[\eta(AZ)W - \eta(Z)AW + 2\eta(Z)W + \eta(W)AZ - \eta(AW)Z - 2\eta(W)Z\,] \quad (4.18)$$

Thus the Distribution $D^{\perp} \oplus \ <\xi>$ is integrable if and only if for any $P[Z,W] = 0$ which proves our assertion.

5. Totally Umbilical Pseudo-Slant Submanifolds of Nearly Quasi-Sasakian Manifolds with Quarter Symmetric Metric Connection

Theorem 5.1 Let M be a totally umbilical pseudo-slant submanifold of a nearly quas–Sasakian manifold $\bar{M}$ with quarter symmetric metric connection. Then at least one of the following statements is true.

(i) $\dim\,(D^{\perp}) = 1$

(ii) $H = \Gamma(\mu)$.

(iii) M is a proper pseudo-slant submanifold.

Proof: Let $z \in \Gamma(D^{\perp})$ and using (2.8), we have

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = \eta(Y)(AX + X) + \eta(X)(AY - Y) - 2g(AX,Y)\xi - 2g(X,Y)\xi$$
$$2(\bar{\nabla}_Z\phi)Z = \eta(Z)(AZ + Z) + \eta(Z)(AZ - Z) - 2g(AZ,Z)\xi - 2g(Z,Z)\xi$$

From last equation, we have

$$(\bar{\nabla}_Z\phi)Z = \eta(Z)AZ - g(AX,Y)\xi - g(Z,Z)\xi$$
$$-A_{VZ}Z - th(Z,Z) - V\nabla_Z Z - th(Z,Z) - nh(Z,Z)$$
$$= \eta(Z)AZ - g(AZ,Z)\xi - g(Z,Z)\xi \quad (5.1)$$

From (2.16) and the tangential component of (5.1), we obtain

$$-A_{VZ}Z - th(Z,Z) = \eta(Z)AZ - g(AZ,Z)T\xi - g(Z,Z)T\xi \quad (5.2)$$

Taking the product by $W \epsilon \Gamma(D^{\perp})$, we obtain

$$g(A_{VZ}Z + th(Z,Z) + \eta(Z)AZ - g(AZ,Z)T\xi - g(Z,Z)T\xi,\ W) = 0$$

It implies that,

$$g(\eta(Z,W),NZ) + g\,(th(Z,Z),W) + \eta(Z)g(AZ,W)$$
$$-g(AZ,Z)g(T\xi,W) - g(Z,Z)g(T\xi,W) = 0 \quad (5.3)$$
$$g(Z,W)g(H,NZ) + g(Z,Z)g(tH,W) + \eta(Z)g(AZ,W) - g(AZ,Z)g(T\xi,W)$$
$$-\eta(Z)g(TZ,W) = 0$$

Since M is totally umbilical submanifold, we obtain

$$g(Z,g(tH,Z)W) + g(Z,g(tH,W)Z) + g(Z,g(AZ,W)\xi - g(Z,g(T\xi,W)AZ)$$
$$-g(Z,g(TZ,W)\xi) = 0 \quad (5.4)$$
$$g(tH,Z)W + g(tH,W)Z + g(AZ,W)\xi - g(T\xi,W)AZ - g(TZ,W)\xi = 0 \quad (5.5)$$

Here tH is either zero or Z and W are linearly dependent vector fields. If tH =0, then dim $\Gamma(D^{\perp}) = 1$, otherwise $H \epsilon \Gamma(\mu)$. since $D_\theta = 0$, M is a pseudo-slant submanifold. Since $\theta = 0$ and $d_1.d_2 = 0$, M is a proper pseudo–slant submanifold.

Theorem 5.2 Let M be a totally umbilical proper pseudo-slant submanifold of a nearly quasi Sasakian manifolds $\bar{M}$ with quarter symmetric metric connection. Then M is an either a totally geodesic submanifold or it is an anti–invariant if H, $\nabla_X^{\perp} H \in \Gamma(\mu)$.

Proof: Since the ambient space is a nearly quasi Sasakian manifold, by using (2.8) we have for any $X \in \Gamma(PM)$,

$$(\bar{\nabla}_X\phi)X = \eta(X)AX - g(AX,X)\xi - g(X,X)\xi + \eta(X)X$$
$$\bar{\nabla}_X\phi X - \phi\bar{\nabla}_X X = \eta(X)AX - g(AX,X)\xi - g(X,X)\xi - \eta(X)\phi X \quad (5.6)$$

Using (2.9), (2.11), (2.12) and (2.16) in (5.6) , we get

$$\nabla_X PX + g(X,PX)H - A_{VX}X + \nabla_X^{\perp}VX = \phi\nabla_X X + g(X,X)\phi H$$
$$+\eta(X)AX - g(AX,X)\xi - g(X,X)\xi - \eta(X)\phi X \quad (5.7)$$

By taking the product with ϕH, we get

$$g(\nabla_X^{\perp}VX,\phi H) = g(V\nabla_X X,\phi H) + g(X,X)\|H\|^2 - g(AX,X)g(V\xi,\phi H) - g(\xi,X)g(VX,\phi H) \quad (5.8)$$

Taking into account (2.10), we get

$$g(\bar{\nabla}_X VX,\phi N) = g(X,X)\|H\|^2 - g(AX,X)g(V\xi,\phi H) - \eta(X)g(VX,\phi H) \quad (5.9)$$

Now for any $X \epsilon \Gamma(PM)$, we obtain

$$\bar{\nabla}_X\phi H = (\bar{\nabla}_X\phi)H + \phi\bar{\nabla}_X H \quad (5.10)$$

In view of (2.10), (2.12), (2.13), (2.21) and (5.10) we obtain

$$[\ \bar{\nabla}_X V = -A_V X + \nabla_X^{\perp}V + \eta(V)\phi X\ ,\ h(X,Y) = g(X,Y)H\ ]$$
$$-A_{\phi H}X + \nabla_X^{\perp}\phi H + \eta(\phi H)\phi X = (\bar{\nabla}_X\phi)H - PA_H X - NA_H X$$
$$+t\nabla_X^{\perp}H + n\nabla_X^{\perp}H \quad (5.11)$$

Taking the product VX to the above equation, we get

$$g(\nabla_X^{\perp}\phi H,VX) = g((\nabla_X\phi)H,VX - g(VA_H X,VX)$$
$$g(\nabla_X^{\perp}\phi H,VX) = g\big((\nabla_X\eta)H + h(tH,X) + VX,VX\big) - g((VA_H X,VX)$$
$$g(A_V X,Y) = g(h(X,Y),V)$$
$$g(VX,VY) = sin^2\theta\{g(X,Y) - \eta(X)\eta(Y)\} \quad (5.12)$$
$$cos^2\theta\ g(X,X)\|H\|^2 + g(X,X)g(V\xi,\phi H) = 0 \quad (5.13)$$

From (5.13), we conclude that $g(X,X)\|H\|^2 = 0$, for any $X \in \Gamma(PM)$, since M is proper pseudo slant submanifold of a nearly quasi Sasakian manifold, we obtain $H = 0$. This tells us that M is totally geodesic in $\bar{M}$.

Theorem 5.3 Let M be a totally umbilical pseudo-slant submanifold of a nearly quasi –Sasakian manifold $\bar{M}$ with quarter symmetric metric connection. Then at least one of the following statements is true.

1). $H \epsilon \mu$.

2). $g(\nabla_{PX}\xi,\ X) = 0$

3). $\eta((\nabla_X P)X = 0$.

4). M is an anti invariant submanifold.

5). If M proper slant submanifold then, $dim(M) \geq 3, X \in \Gamma(PM)$.

Proof: From (2.8) and M is nearly quasi–Sasakian manifold with quarter symmetric metric connection, we have

$$\bar{\nabla}_X \phi X - \phi \bar{\nabla}_X X = \eta(X)AX + \eta(X)X - g(AX,X)\xi - g(X,X)\xi$$

By using (2.9), (2.10), (2.12) and (2.13), we have

$$\nabla_X PX + h(X,PX) - A_{VX}X + \nabla_X^{\perp} VX + \eta(X)VX$$
$$-P\nabla_X X - V\nabla_X X - th(X,X) - nh(X,X)$$
$$= \eta(X)AX + \eta(X)X - g(AX,X)\xi - g(X,X)\xi \qquad (5.14)$$

Tangential component of (5.14) we get

$$\nabla_X PX - P\nabla_X X - th(X,X) - A_{VX}X = \eta(X)AX + \eta(X)X \qquad (5.15)$$

Since M is a totally umbilical pseudo-slant submanifold, then by (2.11) and (2.21) we can write

$$g(A_{VX}X, X) = g(h(X,X),VX) = g(H,VX)g(X,X) = g(g(H,VX)X,X) = 0 \qquad (5.16)$$

If $H \epsilon \Gamma(\mu)$, then from (5.15), we obtain

$$\nabla_X PX - P\nabla_X X = \eta(X)AX + \eta(X)X$$

Taking the product with above by ξ, we get

$$g(\nabla_X PX, \xi) - g(P\nabla_X X, \xi) = \eta(X)g(AX,\xi) + \eta(X)g(X, \xi) = 0 \qquad (5.17)$$

Interchanging X by PX in (5.17), we drive

$$g(\nabla_{PX} P^2X, \xi) = 0 \text{ , implies } g(\nabla_{PX}\xi, P^2X) = 0$$

By using (3.1) we have

$$g(\nabla_{PX}\xi, -cos^2\theta\{X - \eta(X)\xi\} = 0,$$
$$cos^2\theta g(\nabla_{PX}\xi, (X - \eta(X)\xi) = 0.$$

Since M is a proper Pseudo slant submanifold then, we have

$$g(\nabla_{PX}\xi, (X - \eta(X)\xi) = 0.$$

From which

$$g(\nabla_{PX}\xi, X) = \eta(X)g(\nabla_{PX}\xi, \xi) \qquad (5.18)$$

Now we have $g(\xi,\xi) = 1$, taking covariant derivative of above equation with respect to PX for any $X \in \Gamma(PM)$, we obtain

$$g(\nabla_{PX}\xi, \xi) + g(\xi, \nabla_{PX}\xi) = 0 \text{ which implies that } g(\nabla_{PX}\xi, \xi) = 0$$

And then (5.18) gives

$$g(\nabla_{PX}\xi, X) = 0 \qquad (5.19)$$

This proves (2) of the theorem.

Now interchanging X by PX in (5.19) we obtain

$$g(\nabla_{P^2X}\xi, TX) = g\left(\nabla_{cos^2\theta\{X-\eta(X)\xi\}}\xi, PX\right) = 0$$
$$cos^2\theta g\left(\nabla_{(X-\eta(X)\xi)}\xi, PX\right) = 0$$
$$-cos^2\theta g(\nabla_X\xi, PX) + cos^2\theta\eta(X)g(\nabla_\xi\xi, PX) = 0$$

Since, $\nabla_\xi\xi=0$, we obtain

$$cos^2\theta g(\nabla_X\xi, PX) = 0 \qquad (5.20)$$

From (5.20) if $cos\theta = 0, \theta = \frac{\pi}{2}$ then M is an anti variant submanifold. On the other hand $g(\nabla_X\xi, PX) = 0$, that is, $\nabla_X\xi = 0$. this implies that ξ is the Killing vector field on M. If the vector field ξ is not Killing, then we can take at least two linearly independent vectors X and PX to span D_θ.

That is, the $dim(M) \geq 3$.

Example 1. Suppose M is a submanifold of R^7 with coordinates $(x_1, x_2, x_3\ y_1, y_2, y_3, w)$, defined by

$$x_1 = \sqrt{3}\,u \sinh\alpha \frac{\partial}{\partial x_1},\ x_2 = -v\cosh\alpha,\ x_3 = s\sinh z,$$
$$y_1 = v\cosh\alpha,\ y_2 = 2v\cosh\alpha,\ y_3 = -s\sinh z,\ \ t = w$$

Where u , v and z denotes the arbitrary parameters , the tangent bundles of M is spanned by tangent vectors .

$$e_1 = \sqrt{3}\sinh\alpha\frac{\partial}{\partial x_1},\ e_1 = \cosh\alpha\frac{\partial}{\partial y_1} - cosh\alpha\frac{\partial}{\partial x_2} + 2cosh\,\alpha\frac{\partial}{\partial y_2},$$
$$e_3 = \sinh z\frac{\partial}{\partial x_3} - \sinh z\frac{\partial}{\partial y_3},\ e_4 = s\cosh z\frac{\partial}{\partial x_3} - s\cosh z\frac{\partial}{\partial y_3}$$

For the almost contact structure ϕ of R^7 , choosing

$$\phi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i},\ \phi\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j},\ \phi\left(\frac{\partial}{\partial t}\right) = 0,\quad 1 \leq i, j \leq 3,$$

And $\xi = \frac{\partial}{\partial t}$, $\eta = dt$. for any vector field $W = \mu_i \frac{\partial}{\partial x_i} + \nu_j \frac{\partial}{\partial x_j} + \lambda \frac{\partial}{\partial w} \in T(R^7)$ then we have

$$\phi Z = \mu_i \frac{\partial}{\partial y_j} - \nu_j \frac{\partial}{\partial x_i}, \; g(\phi Z, \phi Z) = \mu_i^2 + \nu_j^2,$$
$$g(Z,Z) = \mu_i^2 + \nu_j^2 + \lambda^2, \; \eta(Z) = g(Z, \xi) = \lambda,$$
$$\phi^2 Z = -\mu_i \frac{\partial}{\partial x_i} - \nu_j \frac{\partial}{\partial y_i} - \lambda \frac{\partial}{\partial w} + \lambda \frac{\partial}{\partial t} = -Z + \eta(Z)\xi,$$

For any $i, j = 1,2,3$. it follows that , $g(\phi Z, \phi Z) = g(Z,Z) - \eta^2(Z)$. thus (ϕ, ξ, η, g) is an almost contact metric structure on R^7. Thus we have

$$\phi e_1 = \sqrt{3} \sinh \alpha \frac{\partial}{\partial y_1}, \; \phi e_2 = -\cosh \alpha \frac{\partial}{\partial x_1} - \cosh \alpha \frac{\partial}{\partial y_2} - 2 \cosh \alpha \frac{\partial}{\partial x_2}.$$
$$\phi e_3 = \sinh z \frac{\partial}{\partial y_3} + \sinh z \frac{\partial}{\partial x_3}, \quad \phi e_4 = s \cosh z \frac{\partial}{\partial y_3} + s \cosh z \frac{\partial}{\partial x_3},$$

By direct calculation we can infer $D_\theta = span(e_1, e_2)$ is slant distribution with slant angle

$\theta = cos^{-1}(\frac{1}{\sqrt{6}})$. Since

$$g(\phi e_3, e_1) = g(\phi e_3, e_2) = g(\phi e_3, e_4) = g(\phi e_3, e_5) = 0$$
$$g(\phi e_4, e_1) = g(\phi e_4, e_2) = g(\phi e_4, e_3) = g(\phi e_4, e_5) = 0.$$

e_3 And e_4 are orthogonal to M, $D^\perp = span(e_3, e_4)$ is an anti –invariant distribution. Thus M is 5 – dimensional proper pseudo-slant submanifold of R^7 with its usual almost contact metric structure.

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