Discussion on strong field lensing by wormholes as black hole foils

Amrita Bhattacharya

Kidderpore College, 2,3 Pitamber Sircar Lane
Kolkata-700023, West Bengal, India.

***Abstract-* In this paper it is decided to derive the strong field lensing observables for black hole foils which is considered as the Damour-Solodukhin wormhole and also to examine the range of the deviation parameter λ for which it mimics Schwarzschild black holes in both the classical and quantum level. The lensing observables for the black hole $SgrA^*$ residing in our galaxy can be interpreted to provide an upper bound on $\lambda \sim 10^{-3}$ and until any lower bound is established, all values of λ below the upper bound should be treated equally probable.**

Keywords: **Gravitational lensing, Wormhole, Black hole.**

1 Introduction

Gravitational lensing is one of the very first application of General Relativity that has been ever studied [1]. It helps us to find out the mass of the dark gravitational objects between the source and the observer. This theory is nicely confirmed in the weak field but it is indeed a challenging task in case of the strong gravitational field. For such kind of issue, a black hole or a wormhole could be a form of gravitational lenses.

Virbhadra and Ellis [2] showed that a source behind a Schwarzschild black hole would produce one set of infinite relativistic images on each side of the black hole. The reason for such an interest in gravitational lensing in strong fields is that by the properties of relativistic images it may be possible to investigate the regions immediately outside the event horizon. Moreover, since alternative theories of gravitation must agree with GR in the weak field limit, then, indeed deviation of light rays in strong fields is one of the most promising grounds where the theory of gravitation can be tested in its full form.

Gravitational lensing by a Black hole [3] or a Wormhole [4] is determined when the light rays pass arbitrarily close to the photon sphere in either case. Light rays passing infinitesimally close to the photon sphere winds up a large number of times before escaping away.

Damour and Solodukhin (DS) [5] defined black hole “foils” as objects that mimic some aspects of black holes, while differ in other aspects. This kind of model can be called here as DS wormhole. Wormholes are actually the solutions of Einsteins theory as that of black holes and cannot be ruled out by any kind of experiments. It must have two asymptotically flat mouths connecting two distant regions. A fundamental theoretical distinction between a black hole and a wormhole is that while the former possess an event horizon, the latter has a throat where the gravity is at most. The Null Energy Condition (NEC) and the Weak Energy Condition (WEC) are violated at least at or near the throat due the presence of exotic matter. Despite of having the differences between black hole and wormhole, it is found that many strong field features previously thought of as indicative of a black hole event horizon (i.e. ring-down quasi-normal modes) can be remarkably mimicked by a static wormhole [6-10].

DS wormhole introduces a deviation parameter λ in the Schwarzschild metric converting it in to a black hole foil. The event horizon is then replaced by the wormhole throat $r_{th} = 2GM$. DS found that, if the parameter λ is sufficiently small, i.e. $\lambda \sim e^{-4\pi GM^2}$,then many observational features of a Schwarzschild black hole could be well mimicked by a wormhole in both classical and quantum level. For a distant observer, it is possible to distinguish a wormhole from a black hole when the time scale is sufficiently long that is $\Delta t = 2GMlog\left(\frac{1}{\lambda^2}\right)$, or quantum effect of hawking evaporation again over a long-time scale $\Delta t = 16\pi G^2M^3$. Consider a black hole $SgrA^*$ inside our galaxy started accreting matter 6 billion years ago, it could be a wormhole if $\lambda \ll e^{-10^{15}}$, an incredibly tiny value indeed [5].

The main aim of this paper is to determine the observables for the DS wormhole and examine how small the values of the deviation parameter λ for which it will reproduce the observables for Schwarzschild black hole. In this work, Bozza’s method will be applicable to DS metric for calculating the observables and provide an upper bound of λ.

The paper is organized as follows. In Section 2, we will discuss Bozza’s method for calculating lensing observables in strong field and then in Section 3 it is applied to DS wormhole. Then we will state the numerical comparison in Section 4 and finally the conclusion in section 5.

2 Bozza's method

The method starts with a generic spherically symmetric static spacetime

$$ds^2 = A(x)dt^2 - B(x)dx^2 - C(x)(d\theta^2 + sin^2\theta d\varphi^2) \quad (2.1)$$

The equation

$$\frac{C'(x)}{C(x)} = \frac{A'(x)}{A(x)} \quad (2.2)$$

is assumed to admit at least one positive root and the largest root is called the radius of the photon sphere x_m. The radius of the photon sphere should exceed the horizon radius of the black hole or throat radius of a wormhole as the case may be. A light ray coming from infinity will reach the closest approach distance x_0 from the centre of the gravitating source before emerging in another direction. By the conservation of the angular momentum, x_0 is related to the impact parameter u by

$$u = \sqrt{\frac{C_0}{A_0}} \quad (2.3)$$

The minimum impact parameter is defined by

$$u_m = \sqrt{\frac{C_m}{A_m}} \quad (2.4)$$

where $C_m = C(x_m)$ etc. From the null geodesics, the deflection angle $\alpha(x_0)$ can be expressed as a function of closest approach.

$$\alpha(x_0) = I(x_0) - \pi \quad (2.5)$$

$$I(x_0) = \int_{x_0}^{\infty} \frac{2\sqrt{B}}{\sqrt{C}\sqrt{\frac{C\,A_0}{C_0\,A} - 1}} \quad (2.6)$$

In the weak field limit of deflection, the integrand can be expanded to any order in the gravitational potential and integrated. When we decrease the impact parameter u the deflection angle increases. Decreasing u further bringing the ray infinitesimally closer to the photon sphere will cause the ray to wind up a large number of times before emerging out. Finally, at $x_0 = x_m$, corresponding to the impact parameter $u = u_m$, the deflection angle will diverge and the ray will be captured, i.e., it will wind around the photon sphere indefinitely.

Bozza [3] has defined a new variable

$$z = \frac{y - y_0}{1 - y_0} \quad (2.7)$$

$$y = A(x) \quad (2.8)$$

He has also shown that the divergence is logarithmic for all spherically symmetric metrices and hence the expression of deflection angle becomes

$$\alpha(x_0) = -alog\left(\frac{x_0}{x_m} - 1\right) + b + O(x_0 - x_m) \quad (2.9)$$

Then using the new variable defined by Bozza, the integral (2.6) becomes

$$I(x_0) = \int_0^1 R(z, x_0) f(z, x_0) dz \quad (2.10)$$

$$R(z, x_0) = \frac{2\sqrt{By}}{CA'}(1 - y_0)\sqrt{C_0} \quad (2.11)$$

$$f(z, x_0) = \frac{1}{\sqrt{y_0 - [(1-y_0)z + y_0]\frac{C_0}{C}}} \quad (2.12)$$

Clearly, the function $R(z, x_0)$ is regular for all values of z and x_0, while $f(z, x_0)$ diverges for $z \to 0$, where

$$f(z, x_0) \sim f_0(z, x_0) = \frac{1}{\sqrt{\alpha z + \beta z^2}} \quad (2.13)$$

$$\alpha = \frac{1 - y_0}{C_0 A_0'}(C_0' y_0 - C_0 A_0') \quad (2.14)$$

$$\beta = \frac{(1-y_0)^2}{2C_0^2 {A_0'}^3}\left[2C_0 C_0' {A_0'}^2 + \left(C_0 C_0'' - 2{C_0''}^2\right) y_0 A_0' - C_0 C_0' y_0 A_0''\right] \quad (2.15)$$

where prime denotes differentiation with respect to x.

Now, the angular separation of the image from the lens is $tan\theta = \frac{u}{D_{OL}}$,where D_{OL} is the distance between the observer and the lens [3]. Then the deflection angle equation (2.9) can be written in to a final form

$$\alpha(\theta) \cong -\bar{a}log\left(\frac{u}{u_m} - 1\right) + \bar{b} \quad (2.16)$$

$$u \cong \theta D_{oL} \text{ (Assuming small } \theta) \quad (2.17)$$

$$\bar{a} = \frac{a}{2} = \frac{R(0, x_m)}{2\sqrt{\beta_m}} \quad (2.18)$$

$$\bar{b} = -\pi + b_R + \bar{a}log\frac{2\beta_m}{y_m} \quad (2.19)$$

$$y_m = A(x_m) \quad (2.20)$$

$$\beta_m = \beta_{x_0=x_m} \tag{2.21}$$
$$b_R = \int_0^1 g(z,x_m)\,dz \tag{2.22}$$
$$g(z,x_m) = R(z,x_m)f(z,x_m) - R(0,x_m)f_0(z,x_m) \tag{2.23}$$

Using this, Bozza proposed three strong lensing observables [4]

$$\theta_\infty = \frac{u_m}{D_{OL}} \tag{2.24}$$
$$s = \theta_\infty exp\left(\frac{\bar{b}}{a} - \frac{2\pi}{\bar{a}}\right) \tag{2.25}$$
$$r = 2.5log_{10}\left[exp\left(\frac{2\pi}{\bar{a}}\right)\right] \tag{2.26}$$

where θ_∞ is the asymptotic position approached by a set of images in the limit of a large number of loops the rays make around the photon sphere (θ_∞ is also called the angular radius of the black hole shadow [11]), s is the angular separation between the outermost image and the set of other asymptotic images and r is the ratio between the flux of the first image and the flux coming from all other images.

In our next section we will calculate the strong field lensing coefficients $\{\bar{a},\bar{b},u_m\}$ and the resultant observables (θ_∞,s,r) by applying the previous formulas to DS wormhole. The set $\{\bar{a},\bar{b},u_m\}$ defines the "identity card" [12] of the concerned lens that differs from lens to lens.

3 Application of Bozza's method to DS wormhole

The DS metric is given by

$$ds^2 = (g(r)-\lambda^2)dt^2 - \frac{dr^2}{g(r)}dx^2 - r^2(d\theta^2 + sin^2\theta d\varphi^2) \tag{3.1}$$

where $g(r) \equiv 1 - \frac{2GM}{r}$, the metric differs from the standard Schwarzschild metric only because of the presence of the dimensionless parameter λ. When $\lambda = 0$, we recover a black hole of mass M with an event horizon located at $r = 2GM$. By contrast, when $\lambda \neq 0$ the structure of the spacetime is dramatically different: there is no event horizon, instead there is a throat at $r = 2GM$ that joins two isometric, asymptotically flat regions. This spacetime is an example of Lorentzian wormhole [13].

Then applying the above formulas to DS metric, we get

$$R(z,x_0) = \left(\frac{2M-\lambda^2x_0}{M}\right)\left(\sqrt{\frac{1+2z-z\lambda^2+\lambda^2}{1+2z-z\lambda^2-2\lambda^2}}\right) \tag{3.2}$$
$$f(z,x_0) \sim f_0(z,x_0) = \frac{1}{\sqrt{\alpha z+\beta z^2}} \tag{3.3}$$
$$\alpha = \frac{(\lambda^2x_0-2M)\{3M-x_0(1+\lambda^2)\}}{Mx_0} \tag{3.4}$$
$$\beta = \frac{(\lambda^2x_0-2M)^2\{6M-x_0(1+\lambda^2)\}}{4M^2x_0} \tag{3.5}$$

The radius x_m of the photon sphere can be derived from (2.2) as

$$x_m = \frac{3M}{1+\lambda^2} \tag{3.6}$$

Which gives the expressions in the following forms

$$R(z,x_m) = \left(\frac{2-\lambda^2}{1+\lambda^2}\right)\left(\sqrt{\frac{1+2z-z\lambda^2+\lambda^2}{1+2z-z\lambda^2-2\lambda^2}}\right),$$
$$\alpha_m = \alpha_{x_0=x_m} = 0, \beta_{x_0=x_m} = \frac{(\lambda^2-2)^2}{4(1+\lambda^2)}, y_m = A(x_m) = \frac{1+\lambda^2}{3} \tag{3.7}$$
$$g(z,x_m) = \frac{2}{z}\left[\sqrt{\frac{3\{(1+\lambda^2)-z(\lambda^2-2)\}}{\{(1-2\lambda^2)-z(\lambda^2-2)\}-\{z(\lambda^2-2)+3(1+\lambda^2)\}}} - \frac{1}{\sqrt{1-2\lambda^2}}\right] \tag{3.8}$$

And the minimum impact parameter u_m takes the form

$$u_m = 3\sqrt{3}M(1+\lambda^2)^{3/2} \tag{3.9}$$

Then it follows from the equation (2.18) – (2.19) that the exact coefficients are

$$\bar{a} = \frac{1}{\sqrt{1-2\lambda^2}} \tag{3.10}$$
$$\bar{b} = -\pi + b_R + \bar{a}log\left[\frac{3(\lambda^2-2)^2}{2(1+\lambda^2)^2}\right] \tag{3.11}$$
$$b_R(\lambda) = \int_0^1 g(z,x_m) \tag{3.12}$$
$$g(z,x_m) = \frac{2}{z}\left[\sqrt{\frac{3\{(1+\lambda^2)-z(\lambda^2-2)\}}{\{(1-2\lambda^2)-z(\lambda^2-2)\}-\{z(\lambda^2-2)+3(1+\lambda^2)\}}} - \frac{1}{\sqrt{1-2\lambda^2}}\right]$$

4 Numerical Comparisons

For numerical comparison of lensing observables, we have chosen a massive black hole $SgrA^*$ residing at our galactic centre having $M = 4 \times 10^6 solar\ mass$, $D_{OL} = 8\ kpc, r\ (magnitude) = 2.5 \times log_{10}(r)$. It expresses DS wormhole for $\lambda \neq 0$ and Schwarzschild black hole for $\lambda = 0$.

5 Conclusion

It is very much clear from the table that the strong field coefficients $(\bar{a}, \bar{b})$ and observables $(u_m, \theta_\infty, s, r)$ are very close to the Schwarzschild when $0.001 \leq \lambda \leq 0.05$. As the last two rows show that for $\lambda \sim 10^{-3}$, lensing coefficients and observables exactly determine known Schwarzschild values accurate up to four decimal places [3].

Table 1. Strong field lensing coefficients and observables for DS wormhole

Lens	λ	$\bar{a}$	$\bar{b}$	$u_m \times 10^{12} cm$	$\theta_\infty(\mu as)$	$s(\mu as)$	$r(mag)$
DS wh	0.05	1.0025	-0.4163	3.0876	25.8054	0.0323	6.8048
	0.04	1.0016	-0.4105	3.0835	25.7707	0.0322	6.8109
	0.03	1.0009	-0.4046	3.0802	25.7436	0.0322	6.8157
	0.02	1.0004	-0.4028	3.0779	25.7244	0.0321	6.8191
	0.01	1.0001	-0.4008	3.0765	25.7128	0.0321	6.8212
	0.001	1.0000	-0.4002	3.0761	25.7089	0.0321	6.8218
Sch bh	0	1.0000	-0.4002	3.0761	25.7089	0.0321	6.8218

Among the lensing observables, the best one from the observational point of view is the angular diameter of the shadow $2\theta_\infty$ cast upon the background accretion flow. However, it is evident that the experimental sensitivity is needed to distinguish between DS wormhole and black hole is still far ahead in near future.

Now $\lambda \sim 10^{-3}$ can be interpreted as an upper bound of the parameter. Hence, we conclude that until a lower bound is established all the values of λ below the upper bound should be treated equally probable.

REFERENCES:

[1] A. Einstein, *Science*, 84 (1936) 506.
[2] K.S Virbhadra, G.F.R. Ellis, *Schwarzschild black hole lensing*, *Phys. Rev. D* 62 (2000) 084003.
[3] V. Bozza, *Gravitational lensing in the strong field limit*, *Phys. Rev. D* 66 (2002) 103001
[4] K. K. Nandi, Y.Z. Zhang and A.V. Zakharov, *Gravitational lensing by wormhole*, *Phys. Rev. D* 74 (2006) 024020
[5] T. Damour and S.N. Solodukhin, *Wormholes as black hole foils*, *Phys. Rev. D* 76 (2007) 024016.
[6] R.A Konoplya and A. Zhidenko, *Wormholes versus black holes: quasi-normal ringing at early and late times JCAP* 12 (2016) 043.
[7] V. Cardoso, E. Franzin and P. Pani, *Is the gravitational-wave ringdown a probe of the event horizon? Phys. Rev. Lett.* 116 (2016) 171101.
[8] V. Cardoso and P. Pani, *Tests for the existence of black holes through gravitational wave echoes, Nat. Astron* .1 (2017) 586.
[9] K.K. Nandi, R.N. Izmailov, A.A. Yanbekov and A.A. Shayakhmetov, *Ring down Gravitational waves and lensing observables: how far can a wormhole mimic those of a black hole? Phys. Rev. D* 95 (2017) 104011.
[10] S.H. Volkel and K.D.Kokkotas, *wormhole potentials and throats from quasi- normal modes*, *Class. Quant. Grav.* 35 (2018) 105018.
[11] V. Bozza and L. Mancini, Observing gravitational lensing effect by $SgrA^*$ *with GRAVITY*, A*strophys. J.* 753 (2012) 56.
[12] V. Bozza, *Extreme gravitational lensing by super massive black holes*, *Nuovo Cim*. B 122 (2007) 547.
[13] M. Visser, *Lorentzian Wormholes: From Einstein to Hawking* (AIP, Woodburry, USA,1995), p. 412.